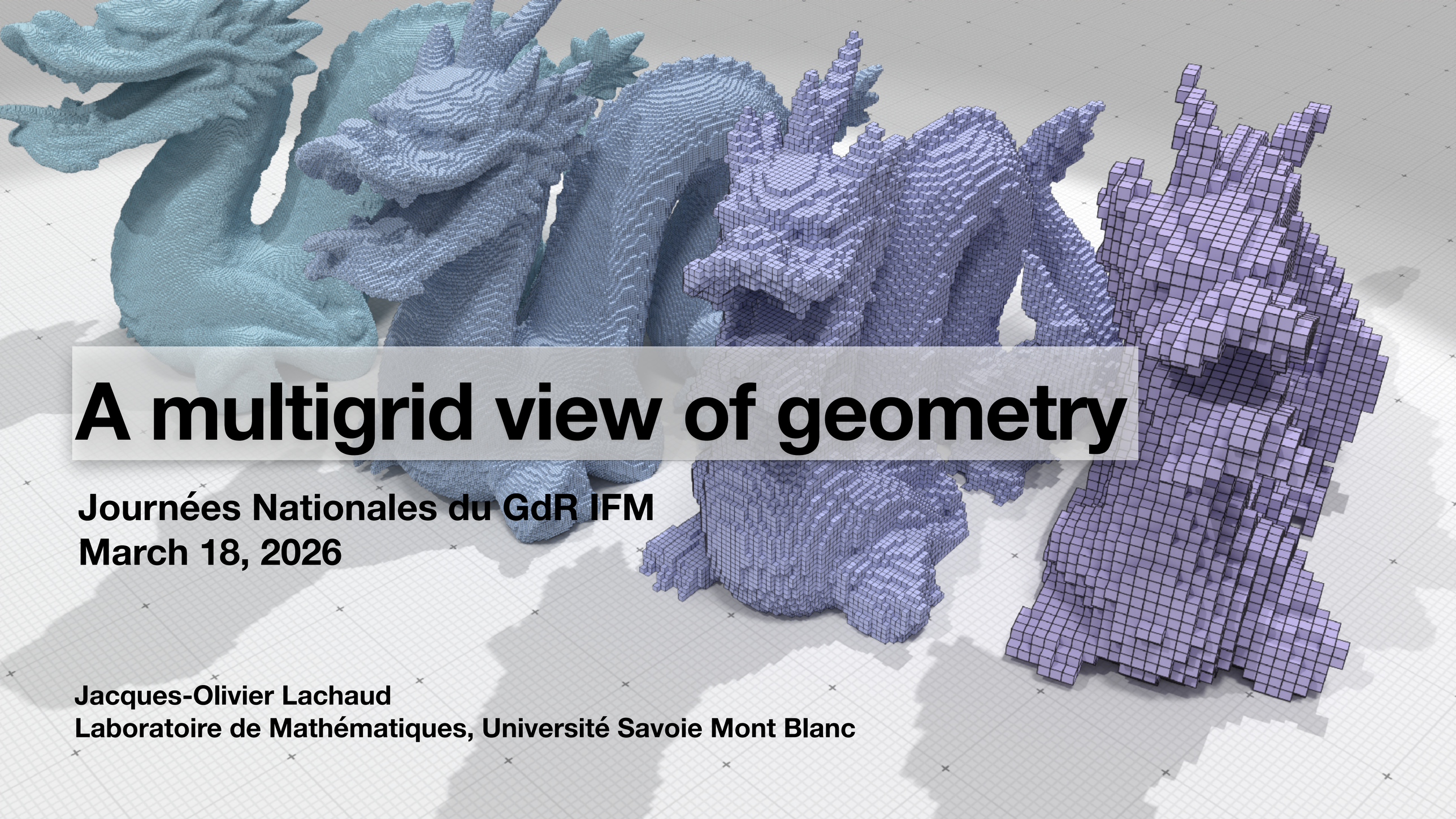


A multigrid view of geometry

Journées Nationales du GdR IFM
March 18, 2026

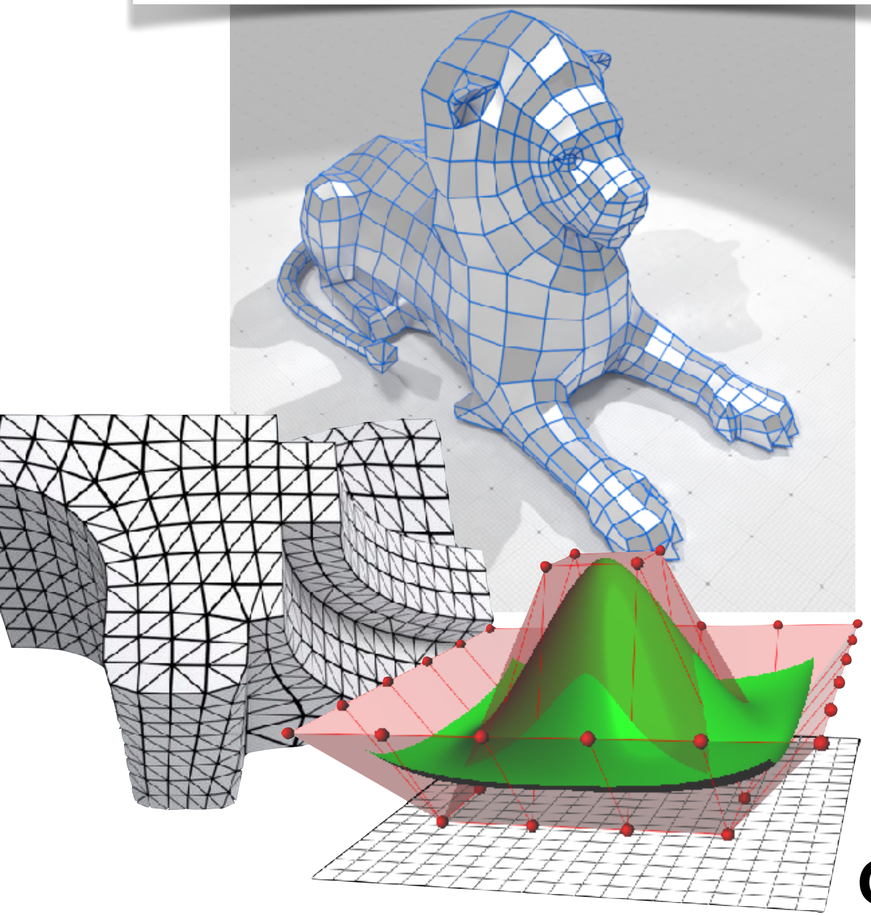
Jacques-Olivier Lachaud
Laboratoire de Mathématiques, Université Savoie Mont Blanc

Outline

- 20 years of digital geometry, context and motivation
- Topology inference and processing
- Geometric inference and multigrid convergence
- Calculus on lattice surfaces
- Recent trends

Digital geometry, context and motivation

Triangle / quad meshes / splines
(Computer-aided design)



Geometric data in computer science

Digital geometry, context and motivation

Triangle / quad meshes / splines
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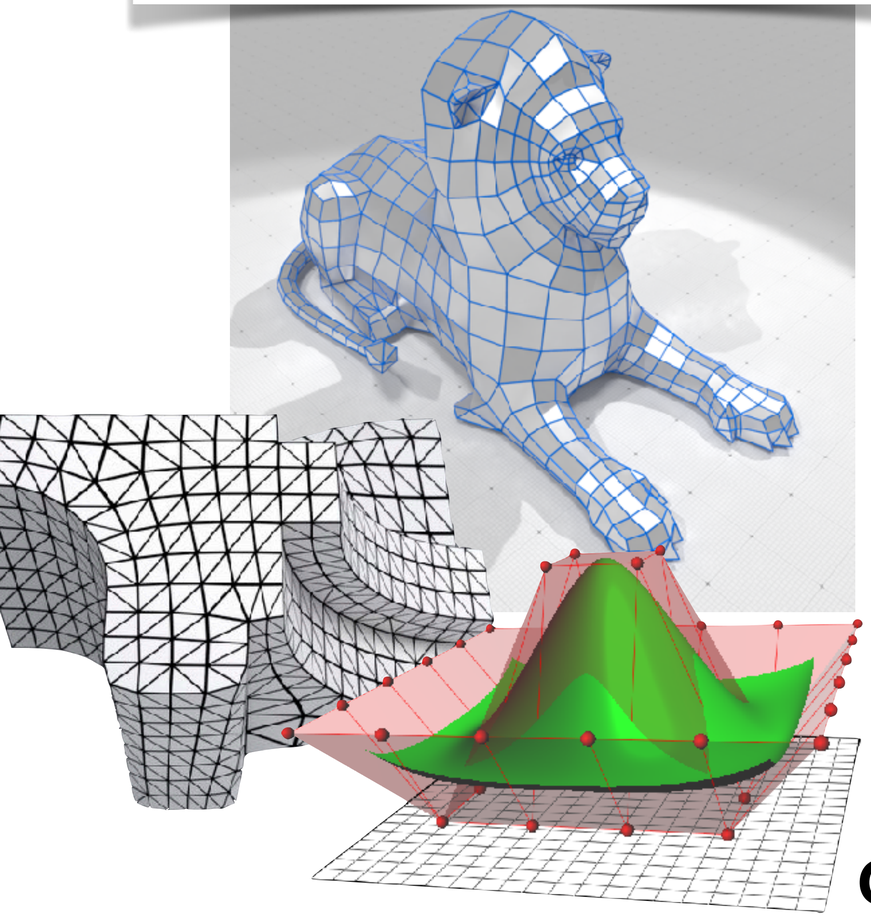
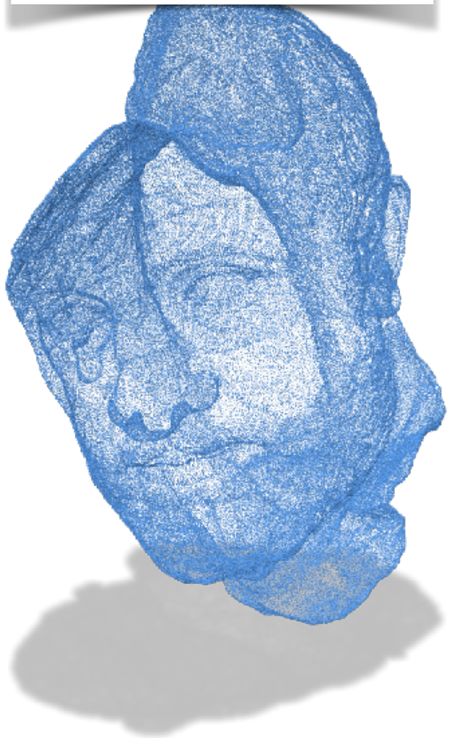


Point clouds
(LiDAR devices)



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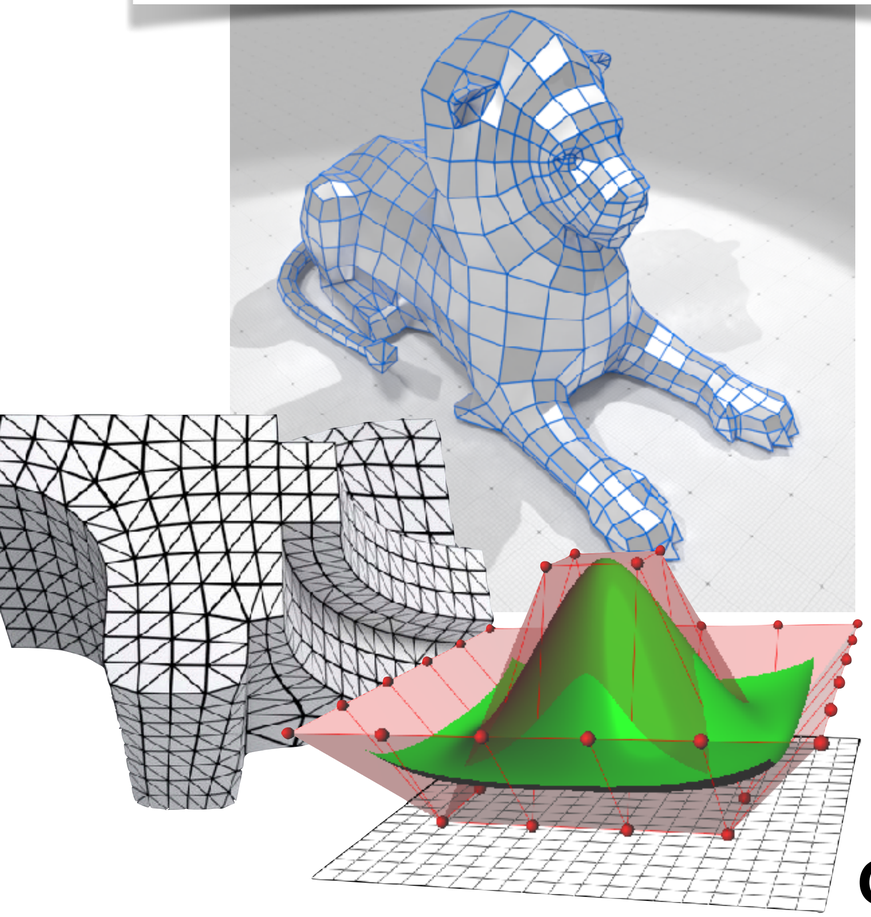

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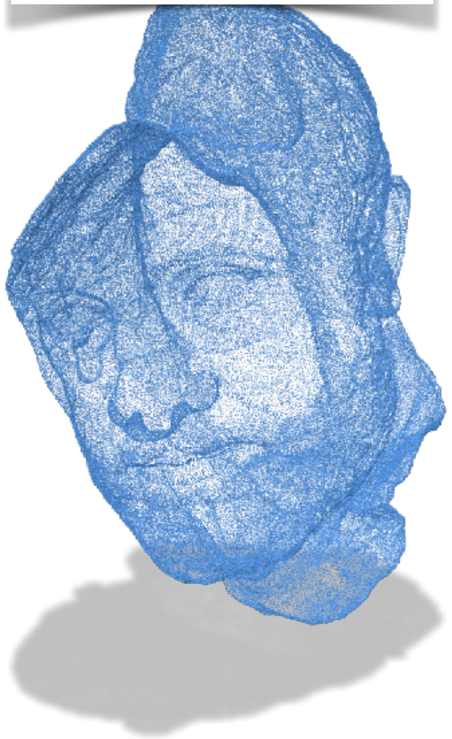
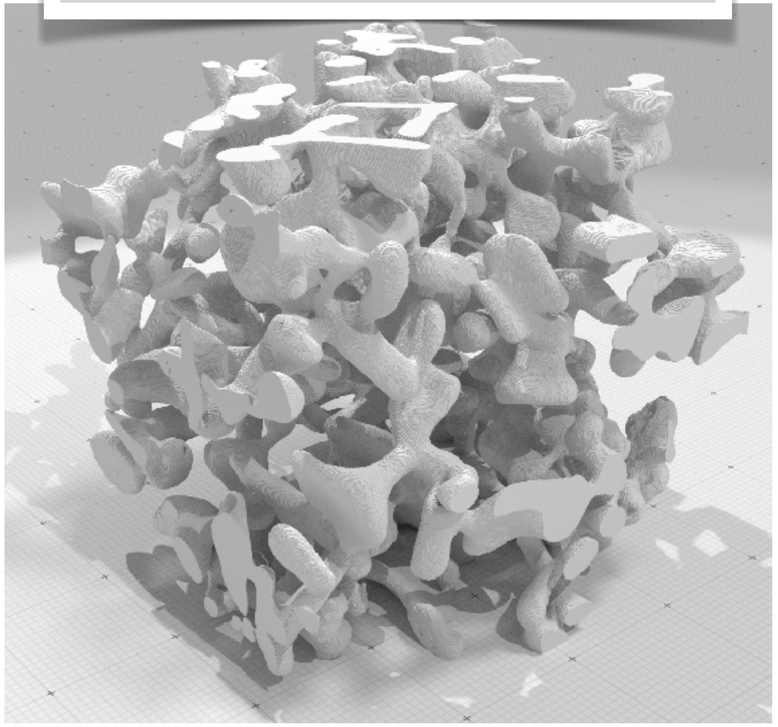


nD Images
(Camera, CT, IRM, PET, …)



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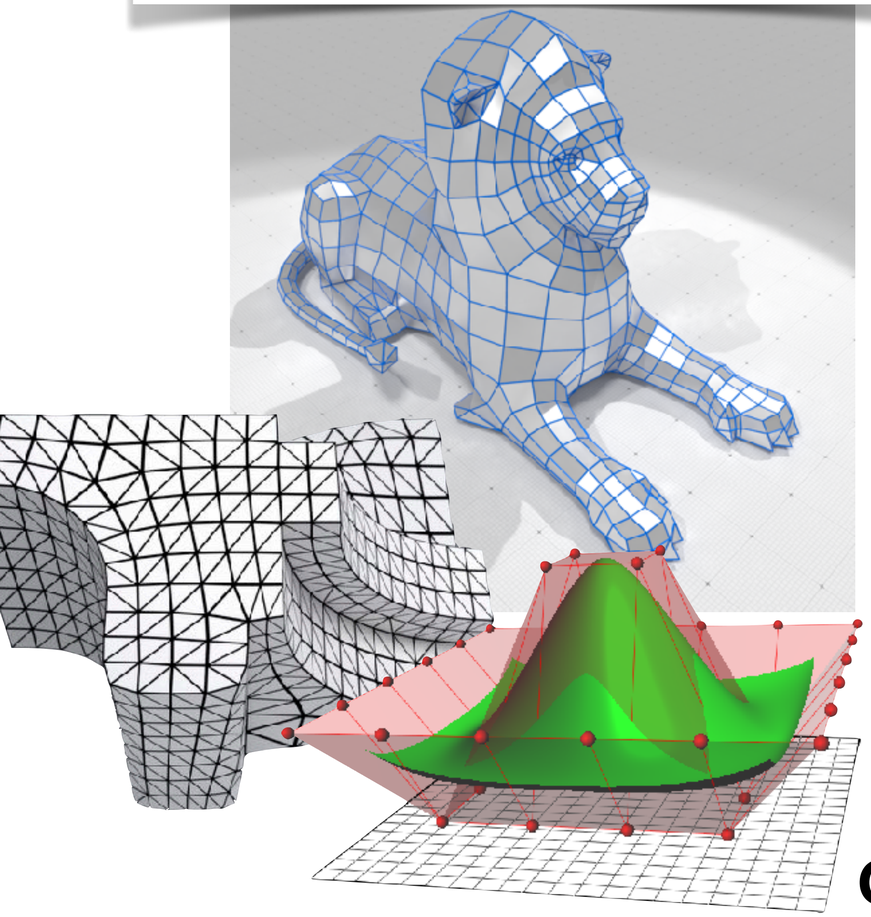

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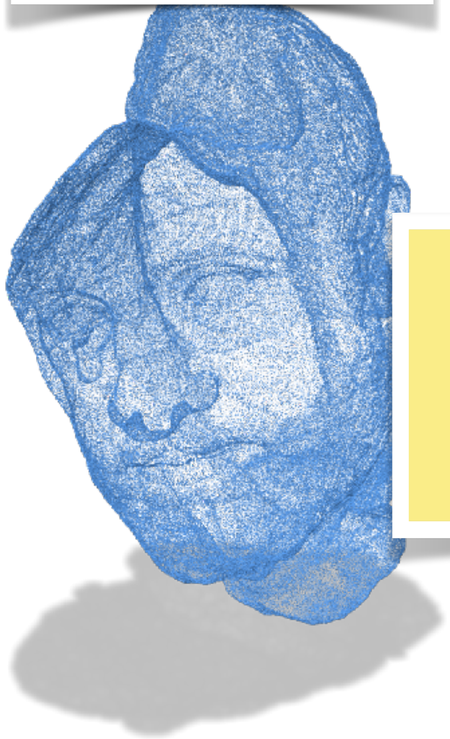

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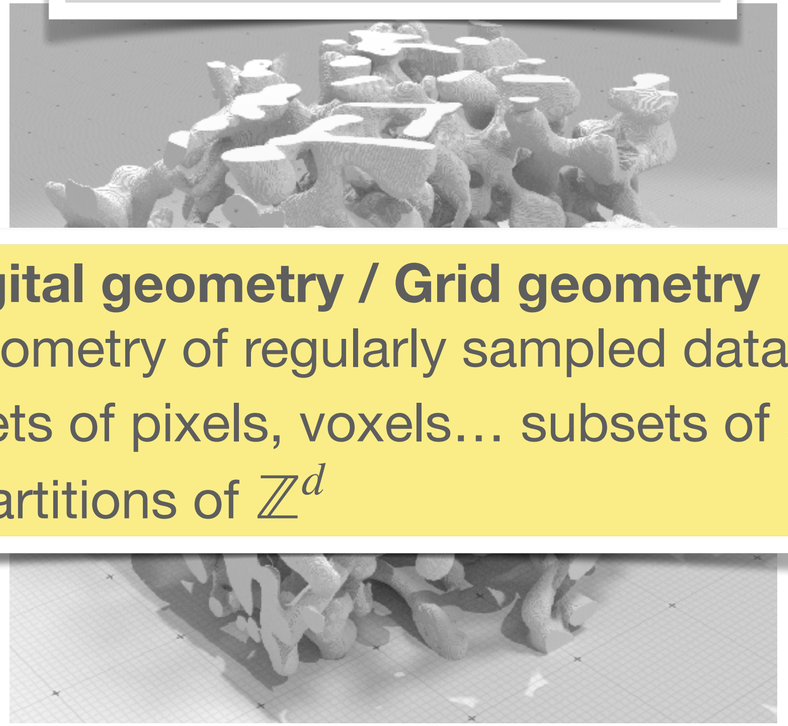

Digital geometry / Grid geometry
= geometry of regularly sampled data

- sets of pixels, voxels… subsets of $\mathbb{Z}^d$
- partitions of $\mathbb{Z}^d$

Geometric data in computer science

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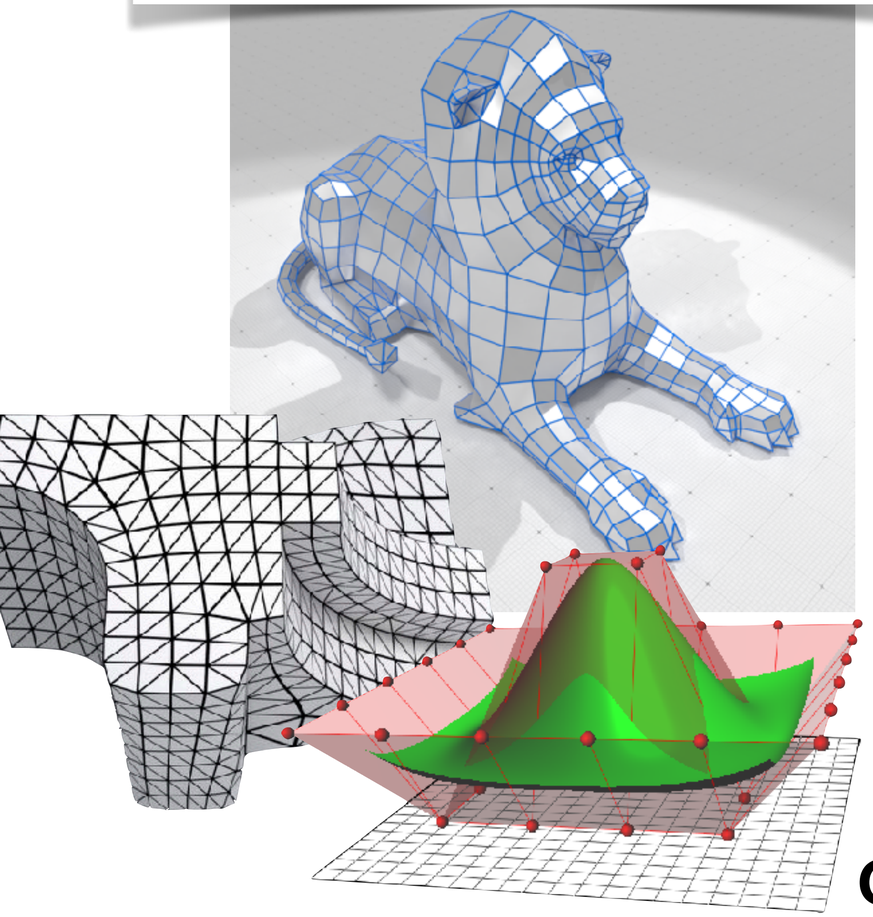

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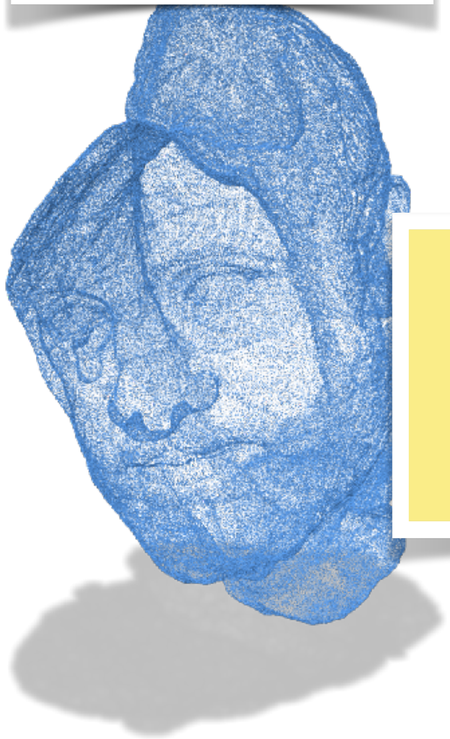

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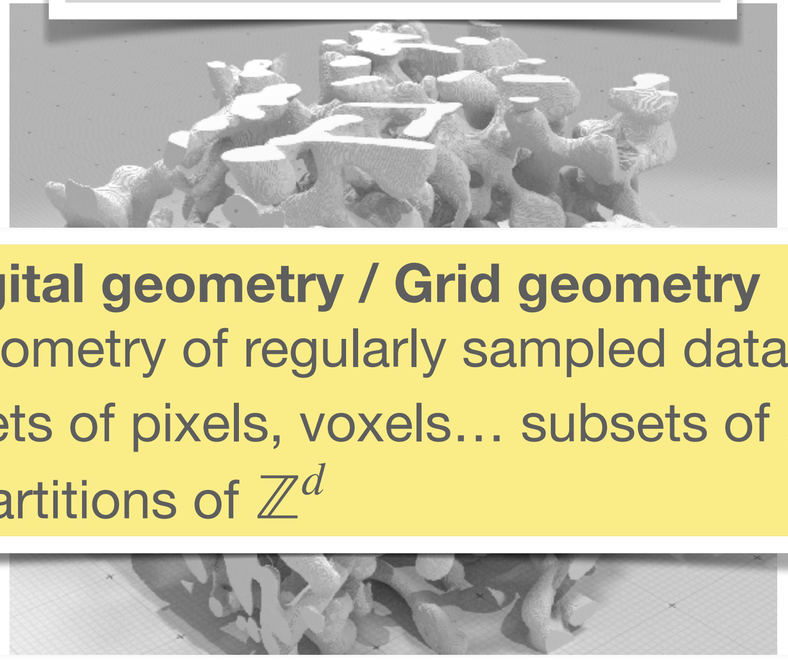

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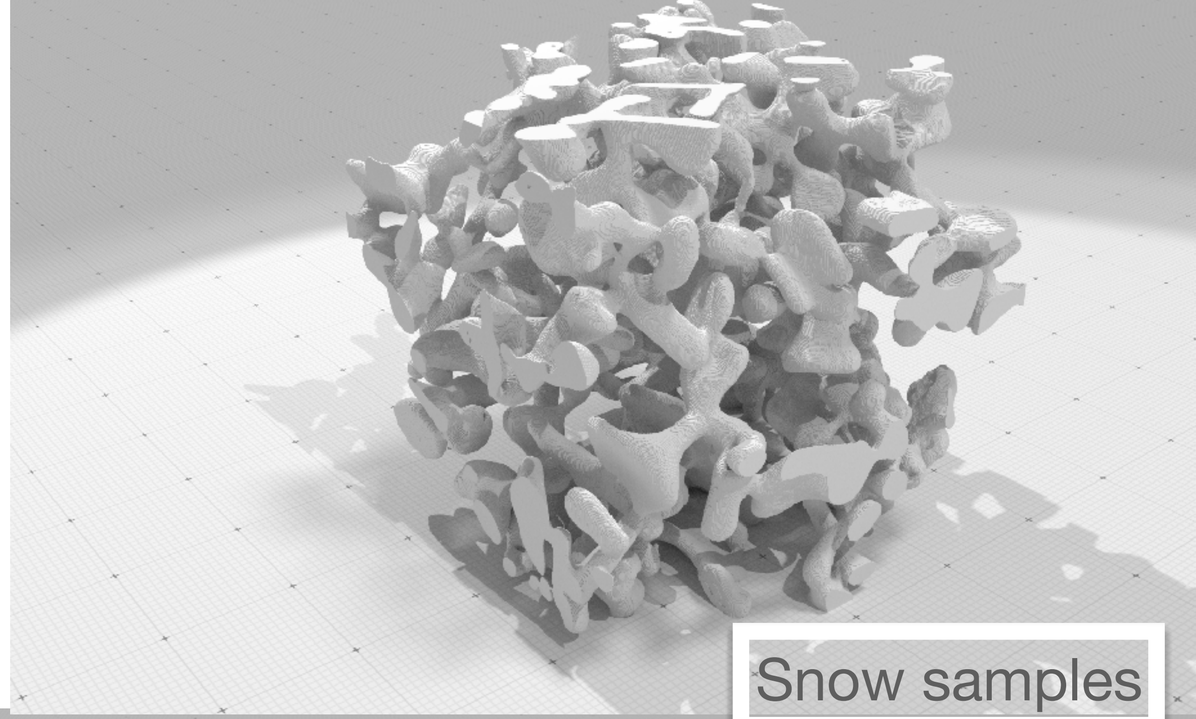
Origins in the **geometry of numbers**
Minkowski, Klein, … [around 1890]

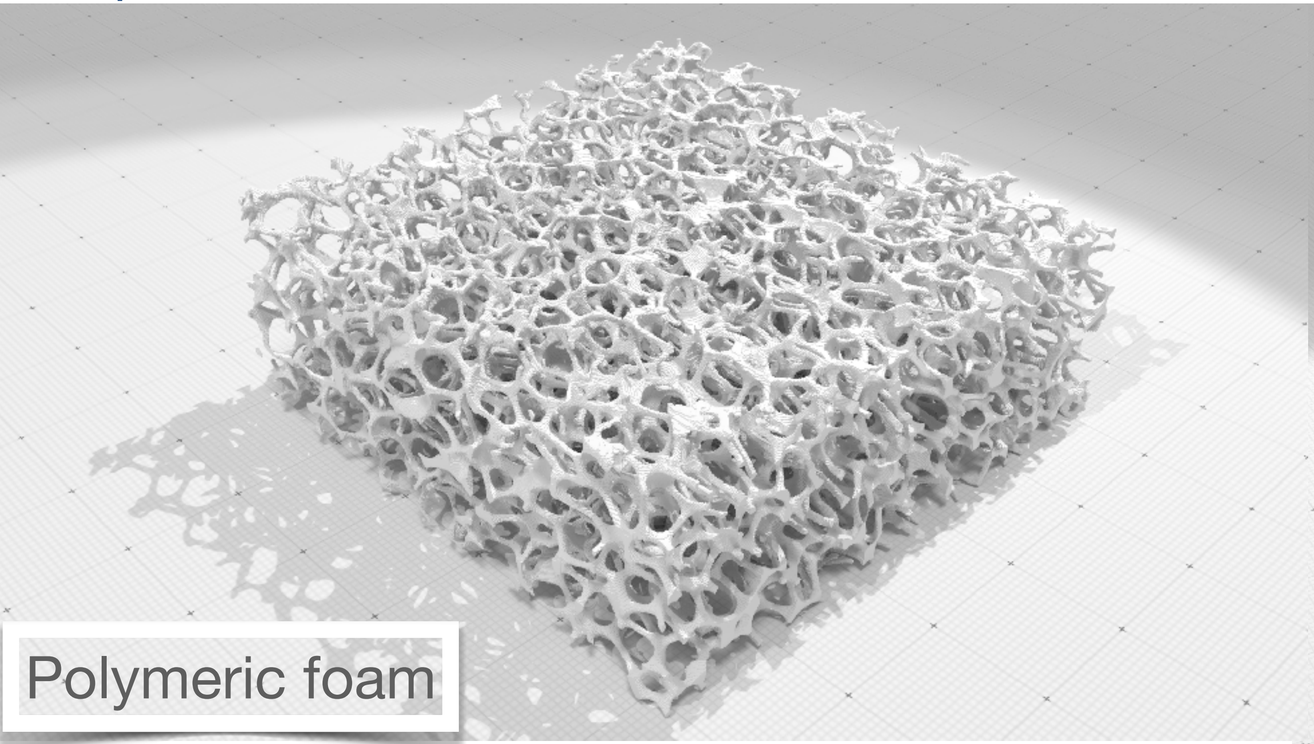
Geometric data in computer science

Motivations (1): devices

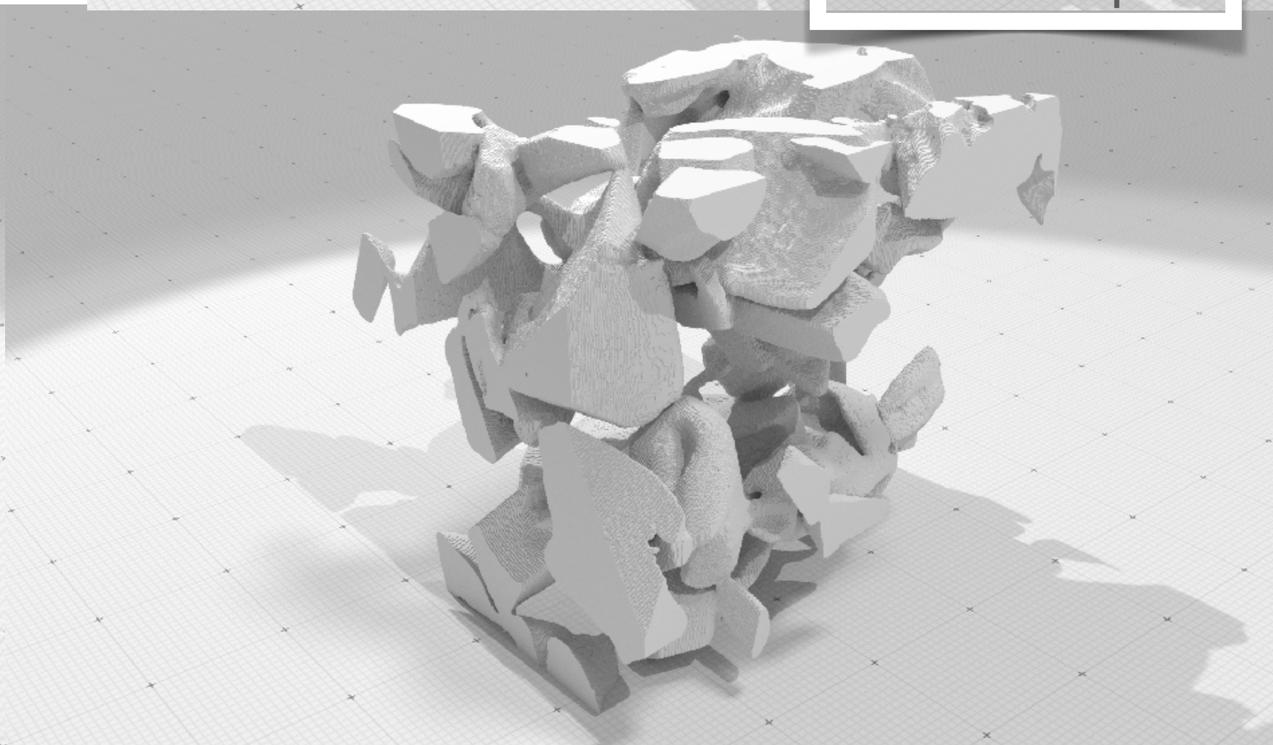
- Micro-tomographic images
  - material sciences
  - medical images
- Process geometry/topology of image subsets and partitions



Snow samples



Polymeric foam



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  - medical images

$$\Rightarrow X \subset \mathbb{Z}^d$$

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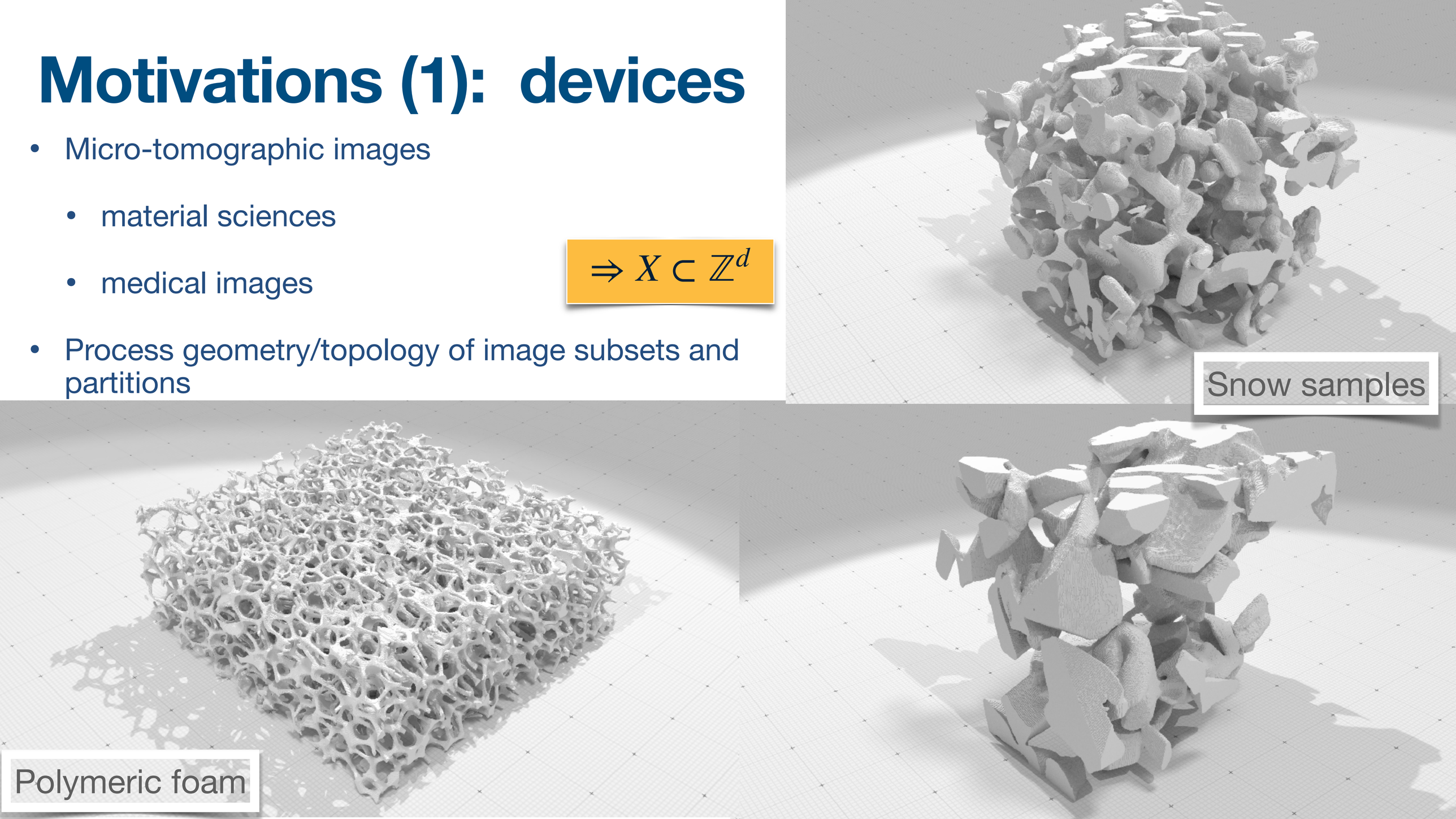
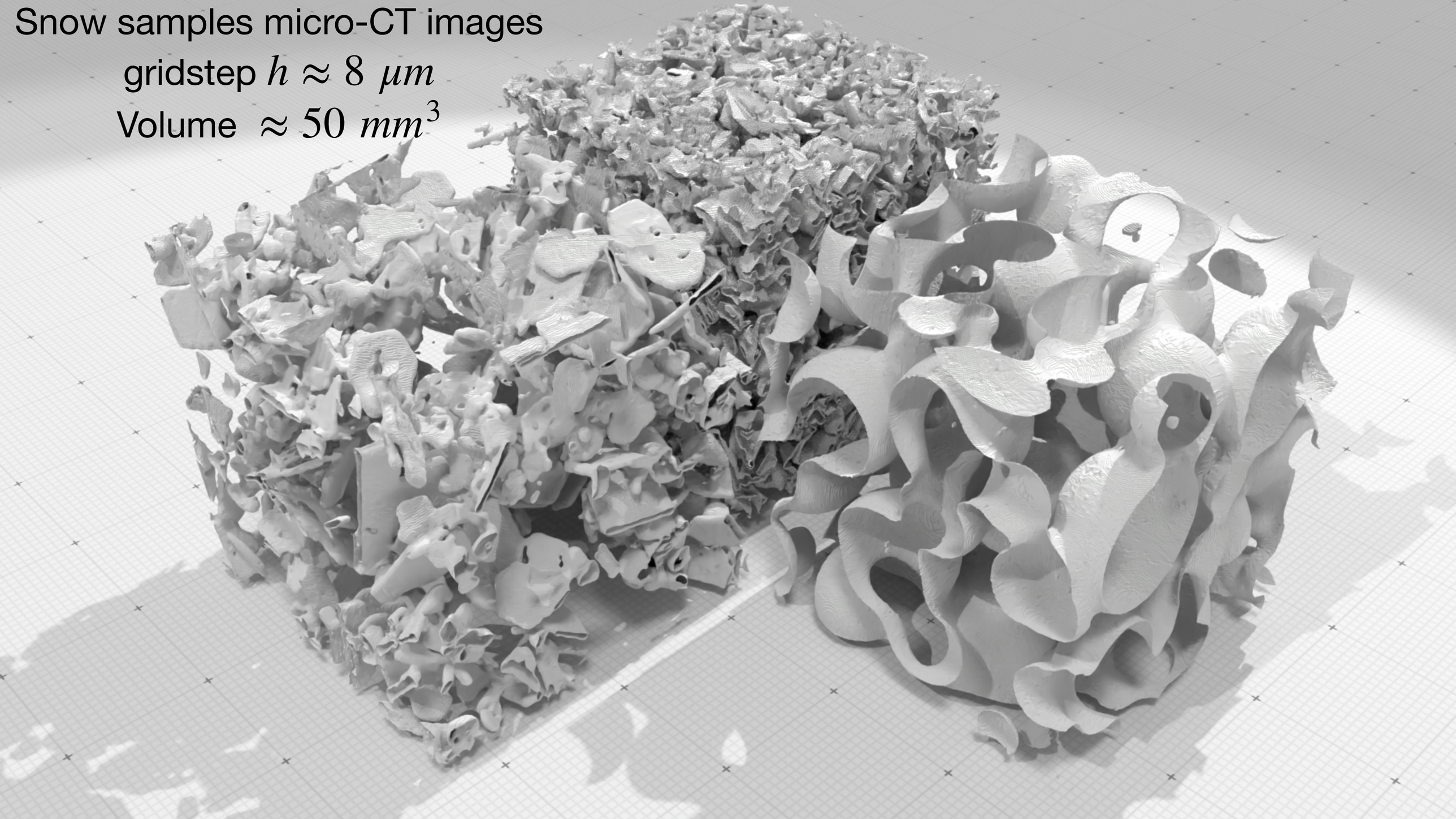


Snow samples micro-CT images

gridstep $h \approx 8\ \mu m$

Volume $\approx 50\ mm^3$



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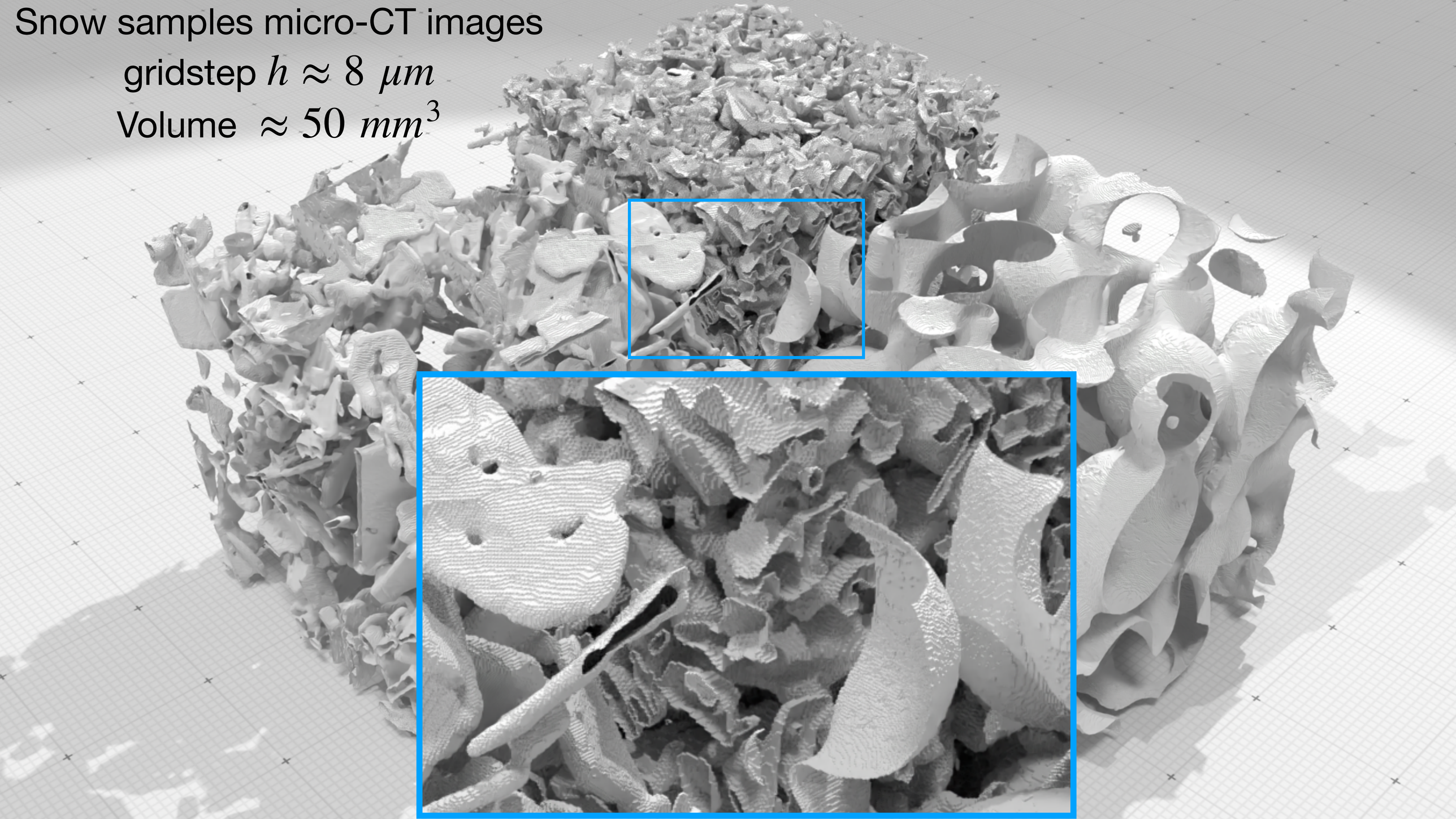

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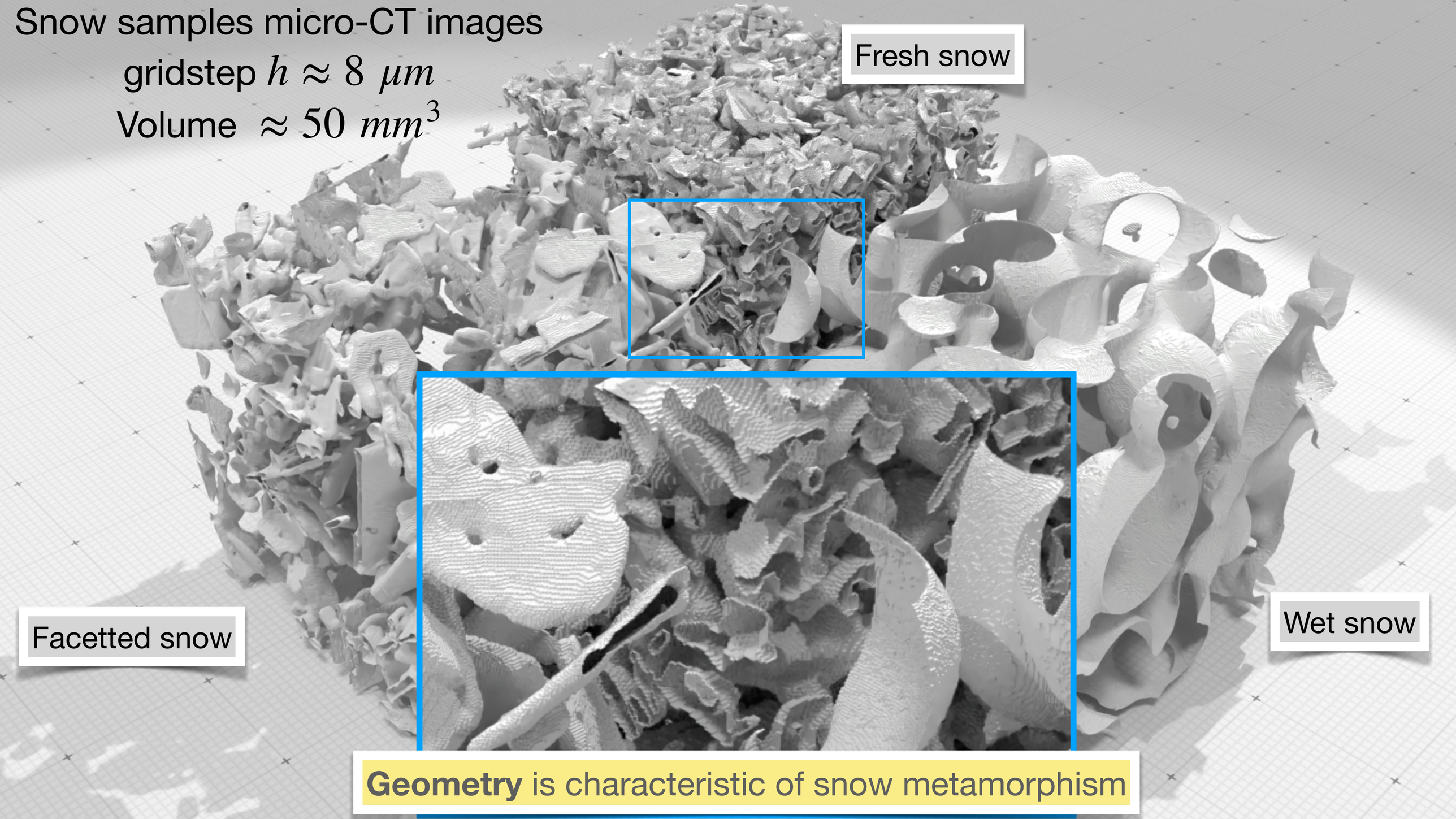
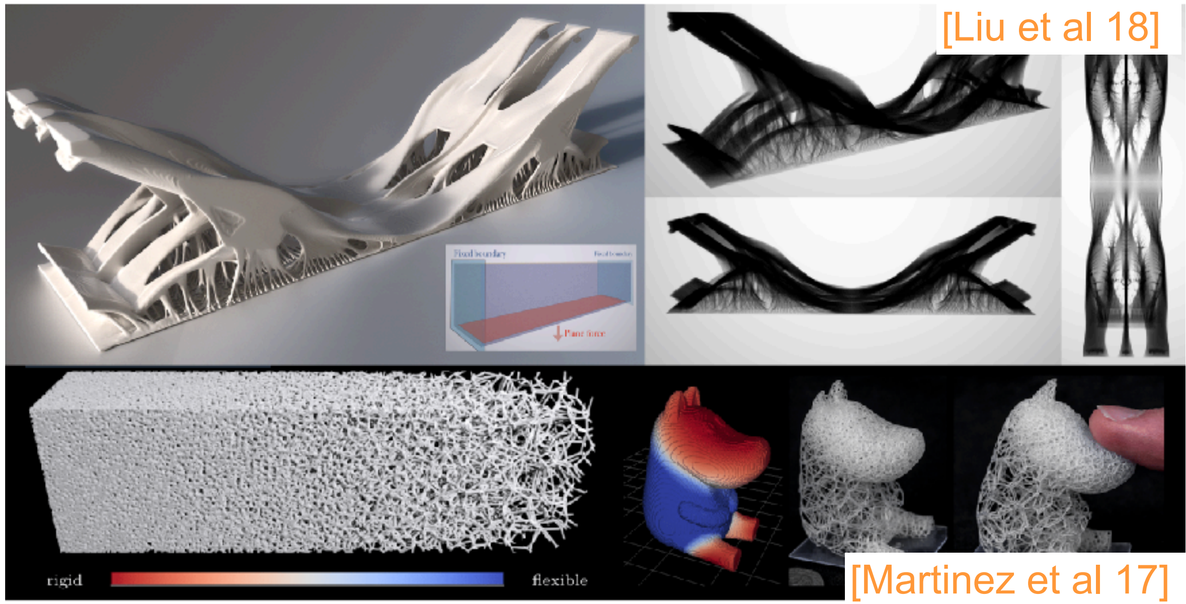


Geometry is characteristic of snow metamorphism

Motivations (2): $\mathbb{Z}^d$ as an efficient modelling space

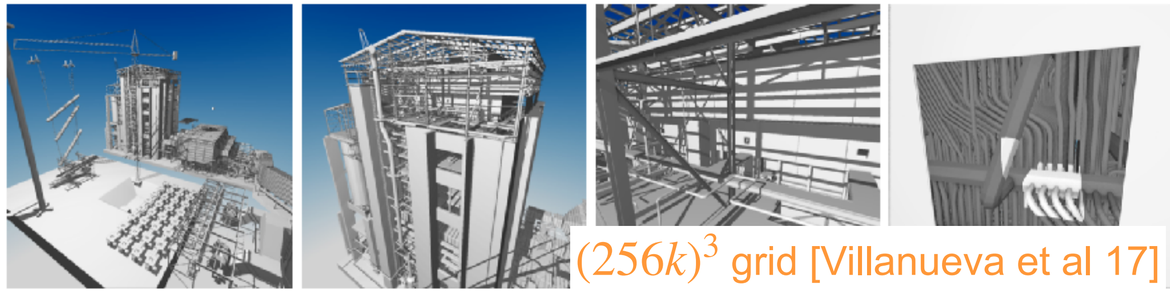
- Shape optimization / fabrication
- Geometric deep learning
- As a proxy or an intermediate representation

  light transport simulation, booleans, medial axis, distance fields, multiple interfaces/objects tracking in a simulation loop…



[Liu et al 18]

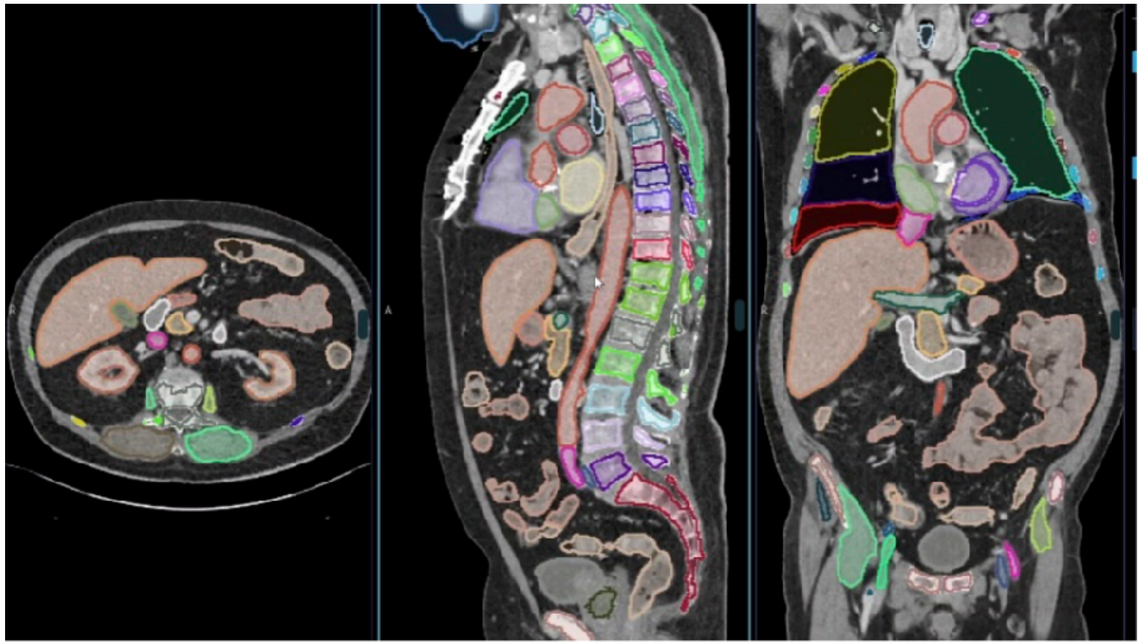
[Martinez et al 17]



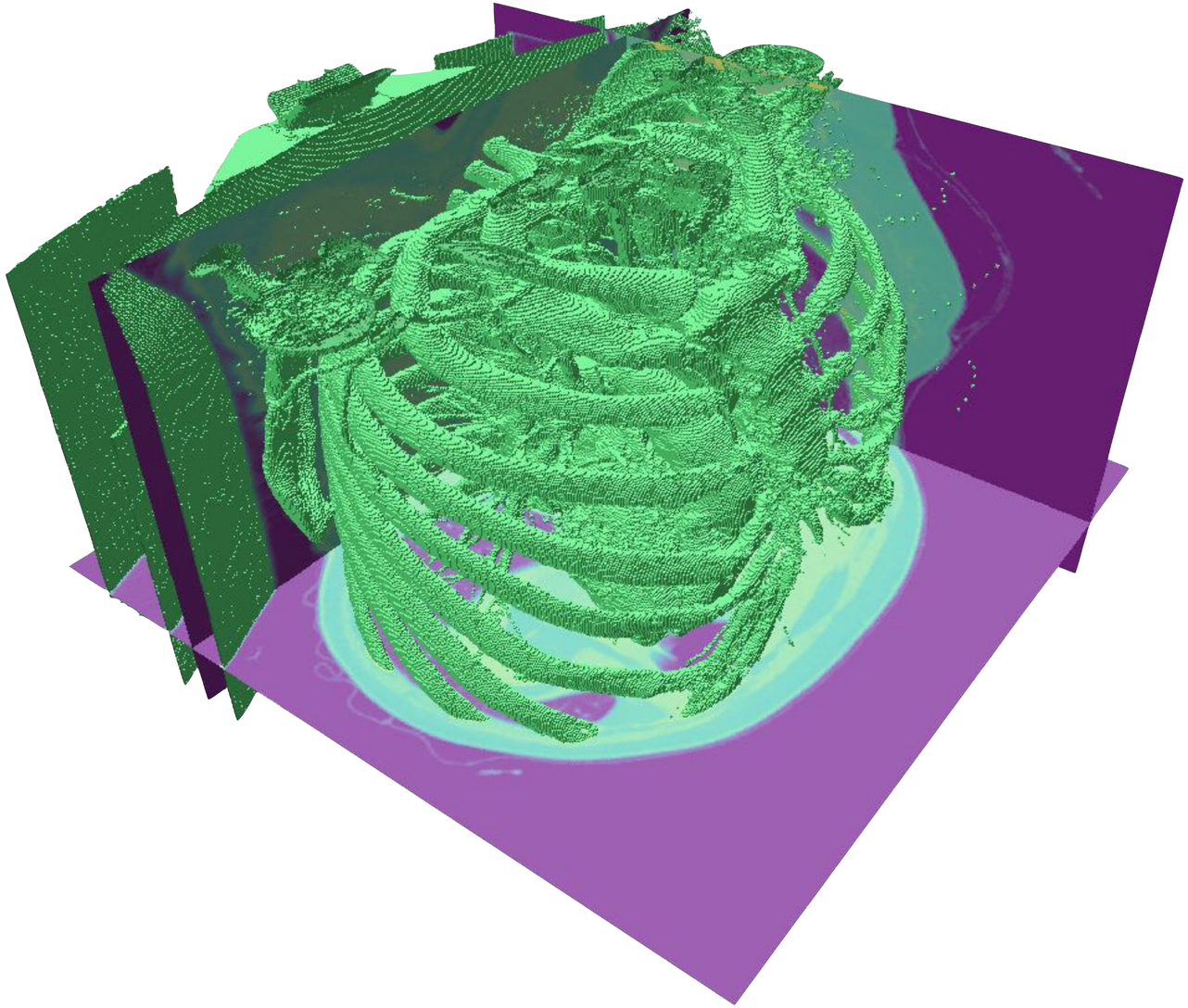
$(256k)^3$ grid [Villanueva et al 17]

***Focus**: characteristic functions / labelled images / level sets / …*

Motivations (3): $\mathbb{Z}^d$ is the support for filtering and segmentation



NVIDIA VISTA 3D



- Many 3D images devices: IRM, CT, confocal microscopy, PET
- Extract objects of interest
- Measure and compare them intra- and extra-
- Diagnostic, prognostic, planning, etc

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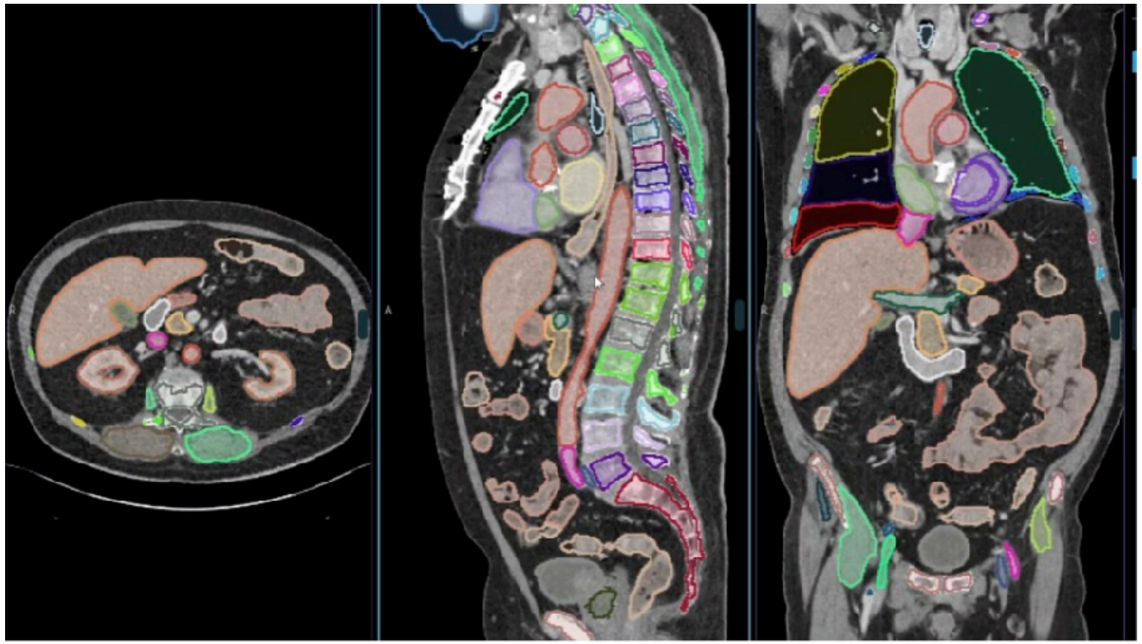

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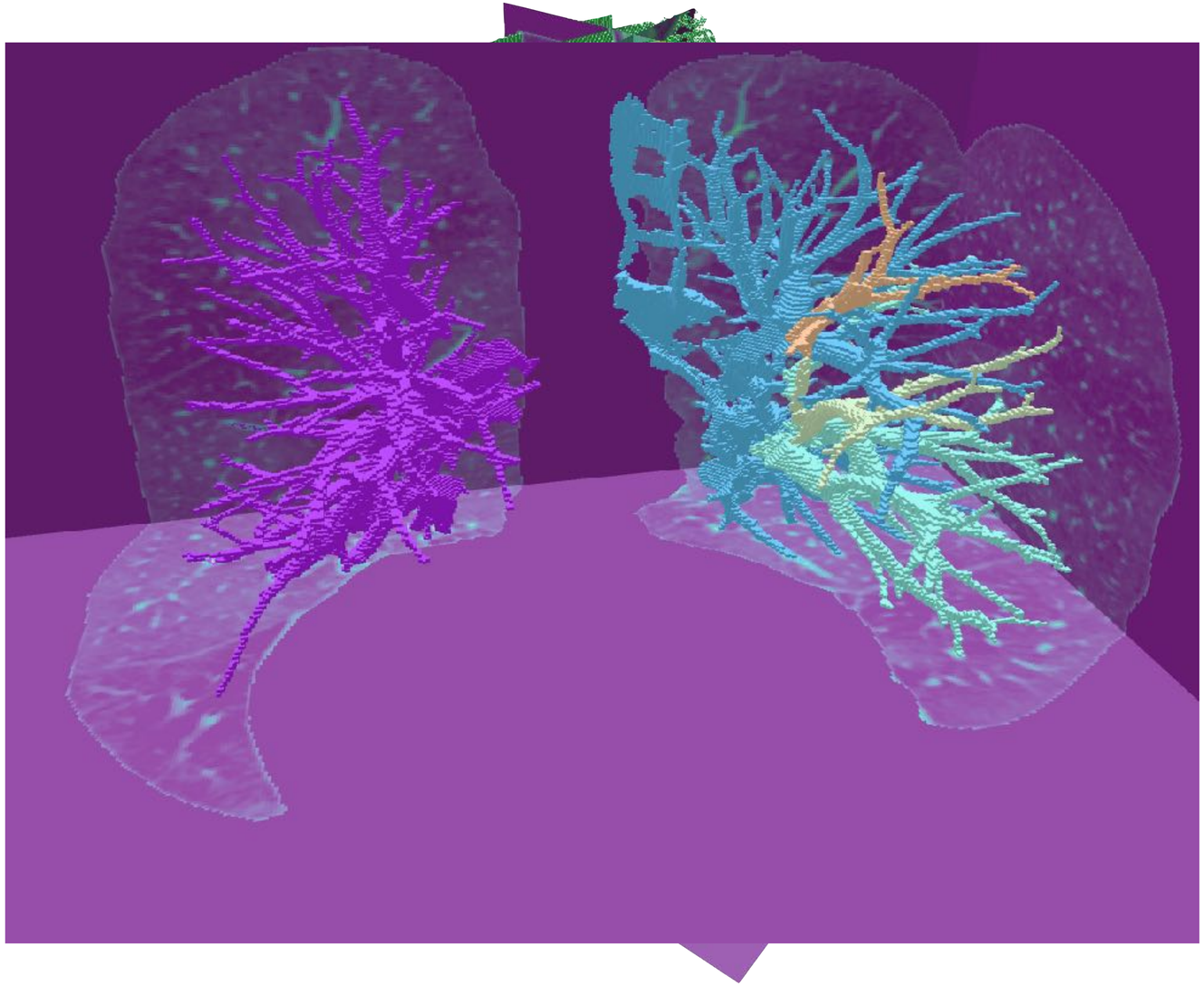

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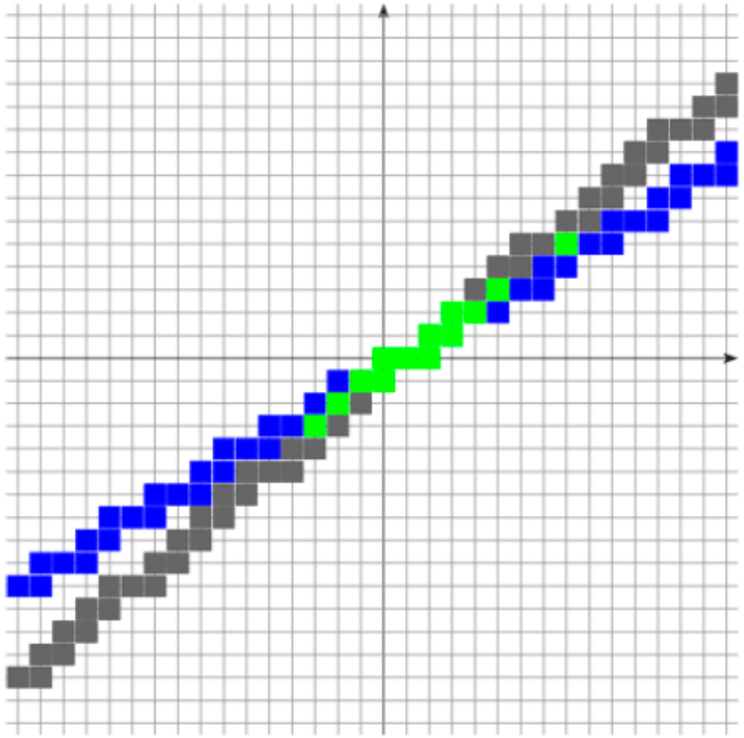
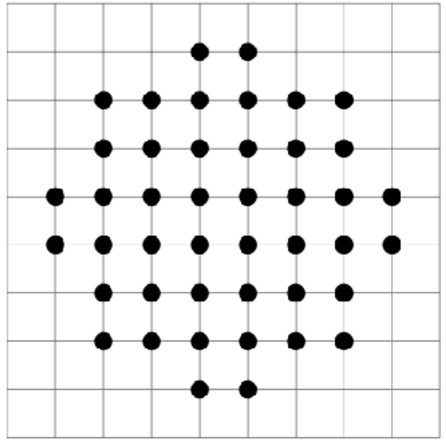
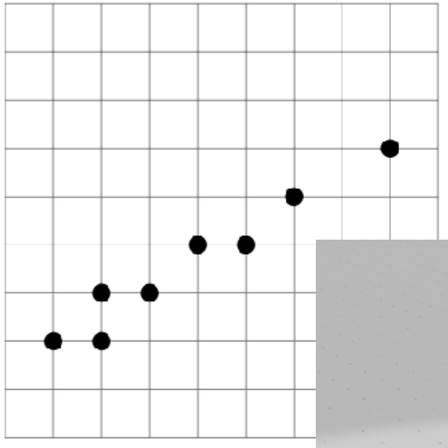
Motivations (4): geometry of $\mathbb{Z}^d$ is weird… and fun !



Intersection of straight lines
are not reduced to a point

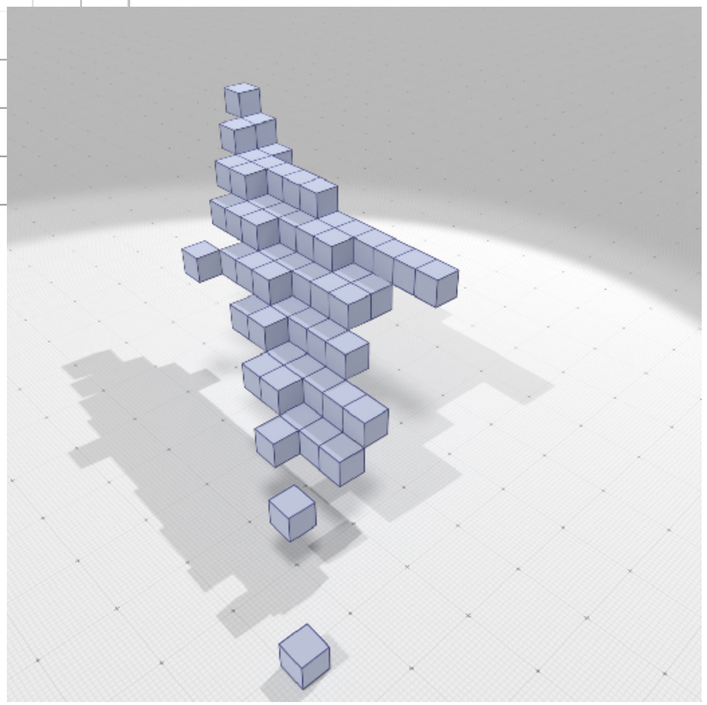
Convexity is not applicable. Digital convexity is… special !





All these set of points
are digitally convex !

i.e. $X = \mathrm{CvxH}(X) \cap \mathbb{Z}^d$



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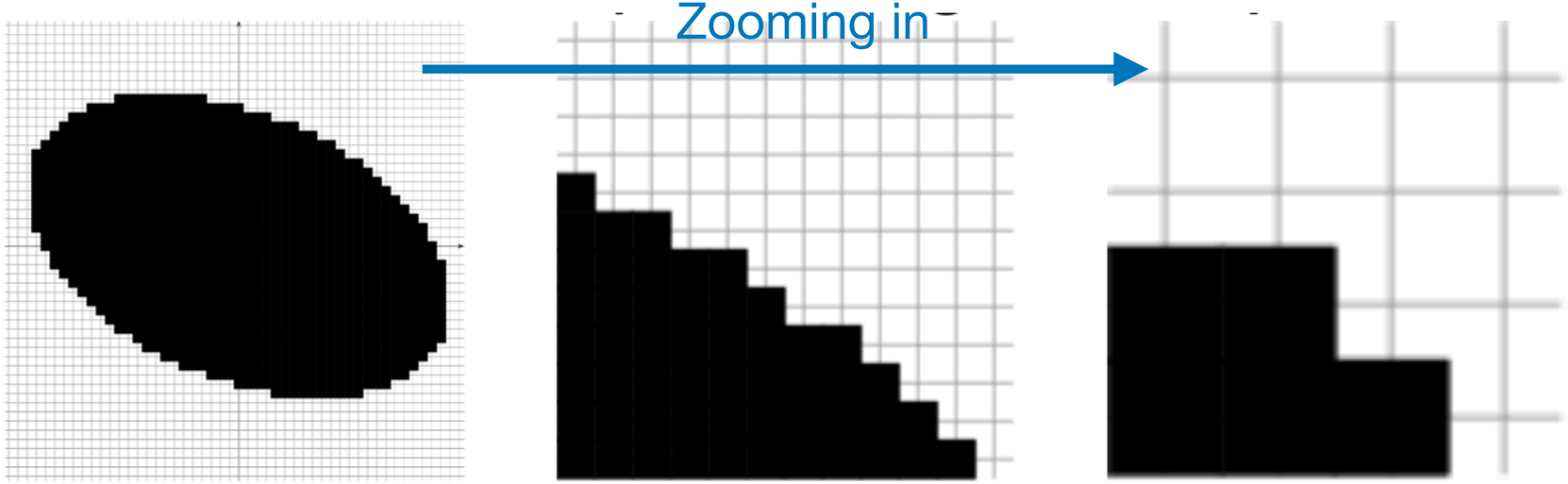
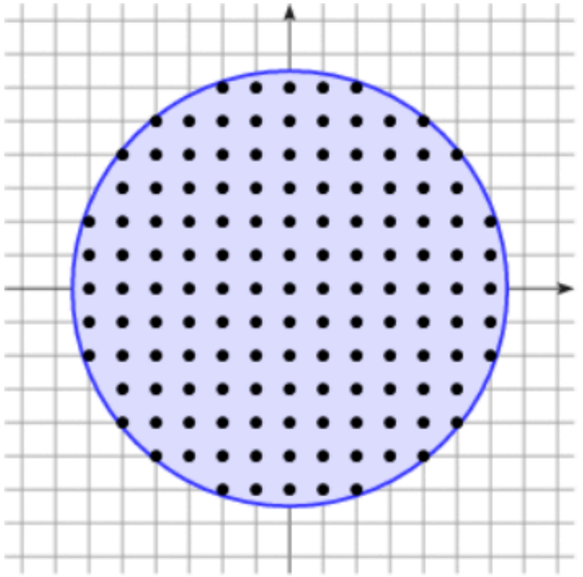
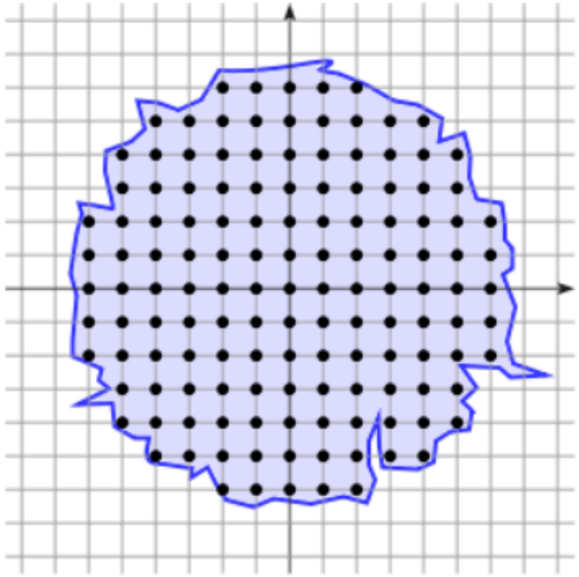


No interesting differential geometry !

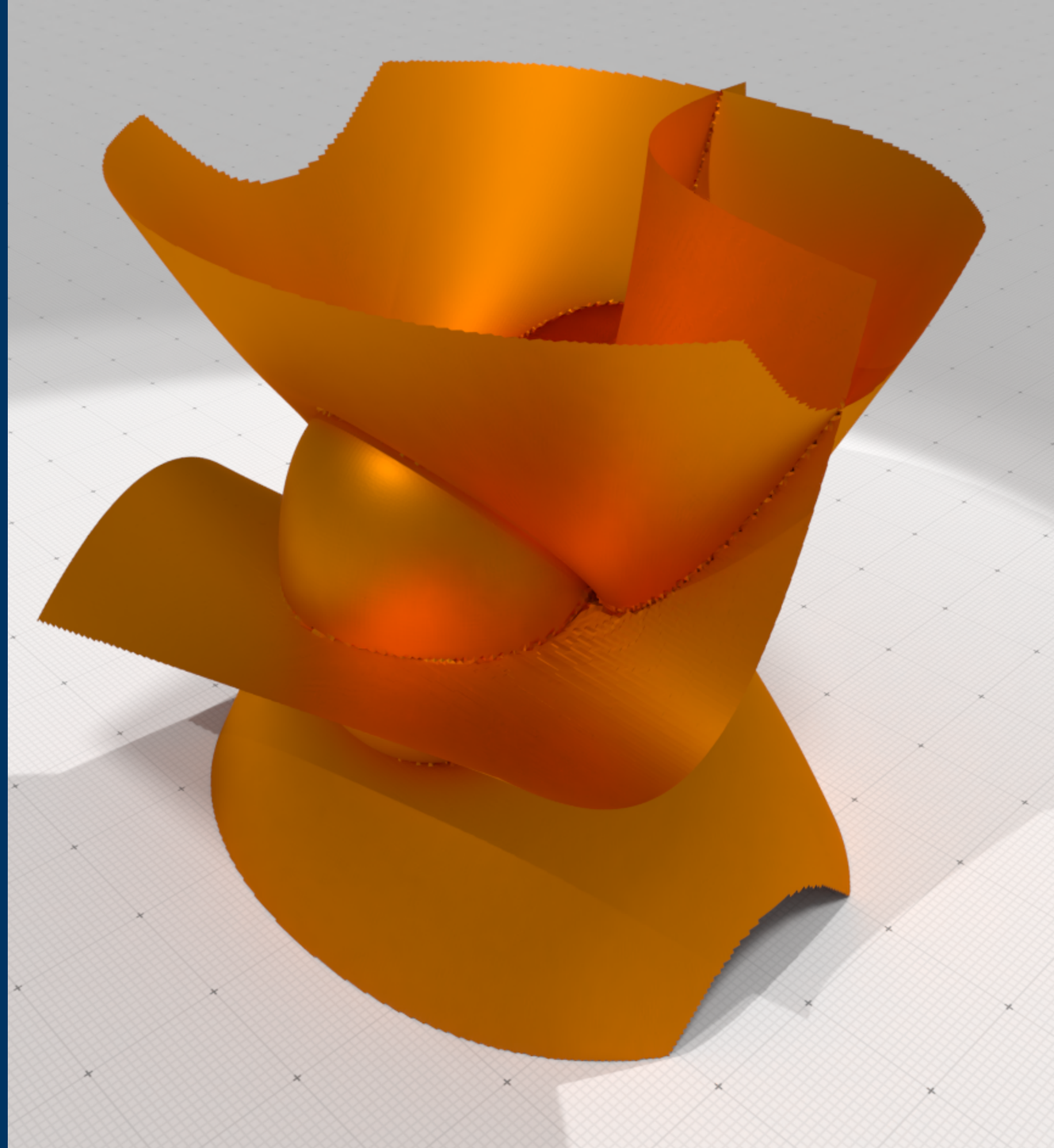


?



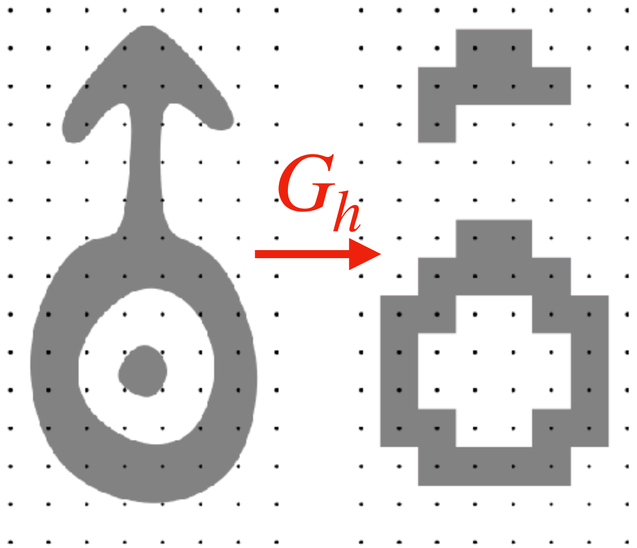
Infinitely many shapes have the same digitisation !

Topology inference and proximity in $\mathbb{Z}^d$



Is topology preserved through digitisation ?

1. How volumes are preserved ?



Volume = d-manifold in $\mathbb{R}^d$

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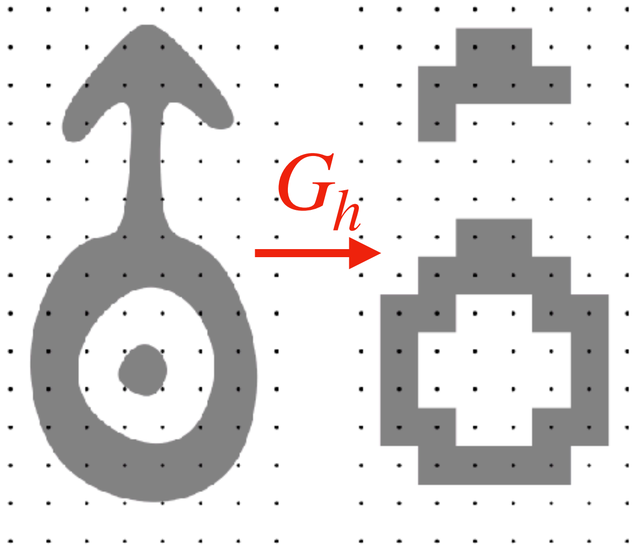
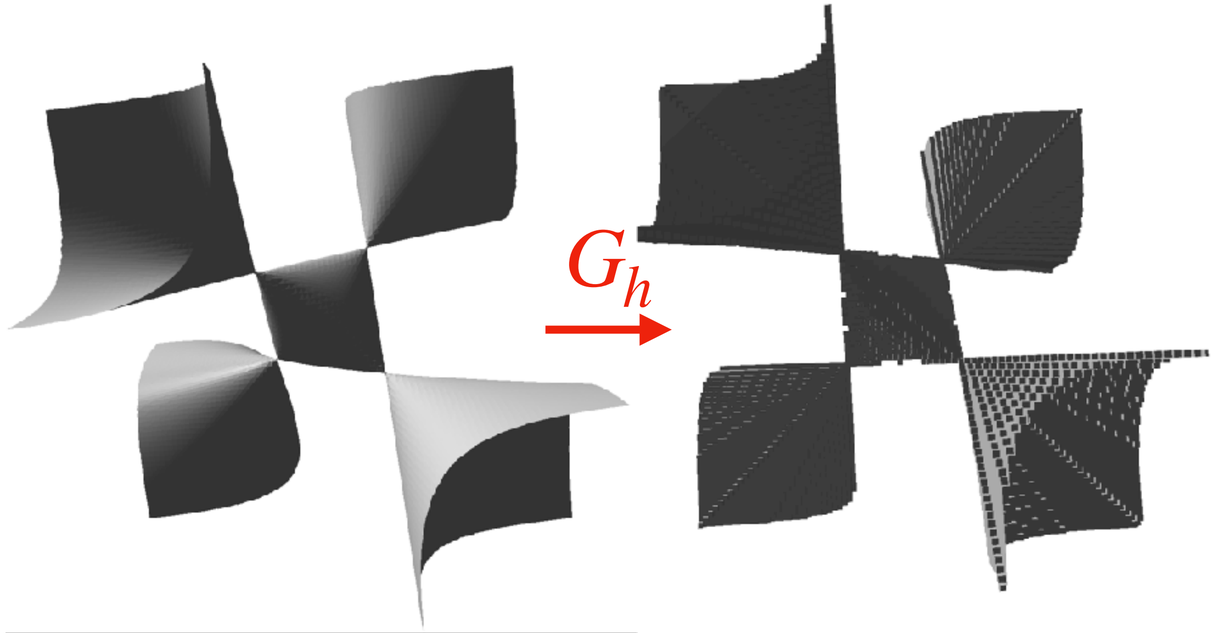


Volume = d-manifold in $\mathbb{R}^d$

2. How surfaces are preserved ?

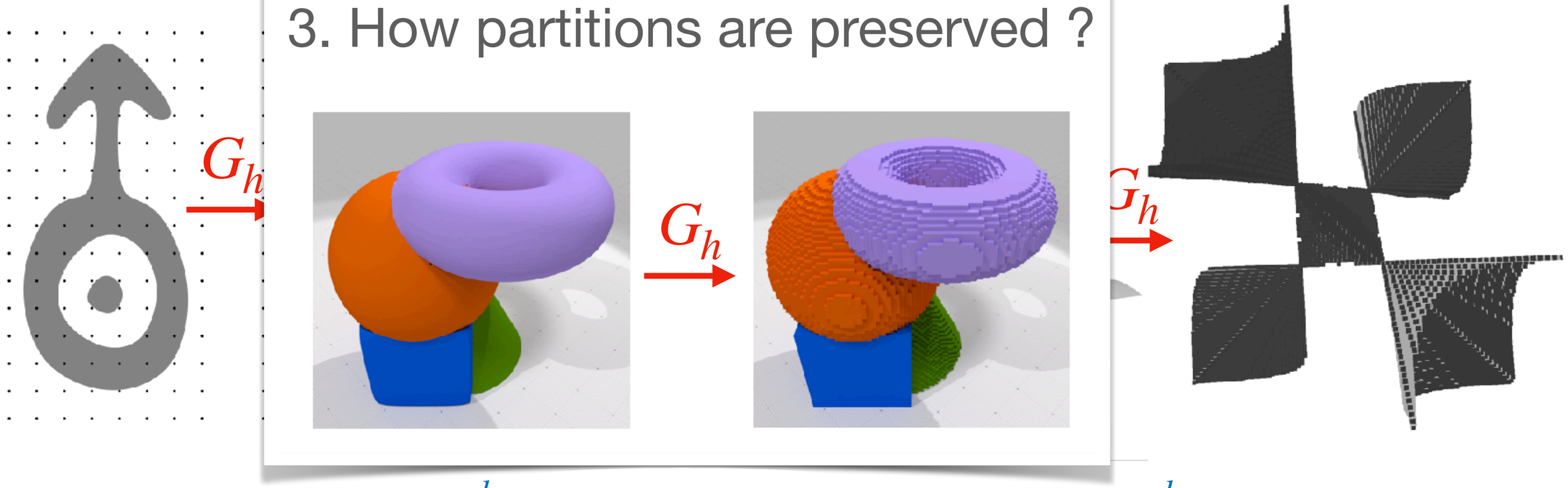


k-manifold in $\mathbb{R}^d$, $k \leq d$

Is topology preserved through digitisation ?

1. How volumes are preserved ? 2. How surfaces are preserved ?

3. How partitions are preserved ?



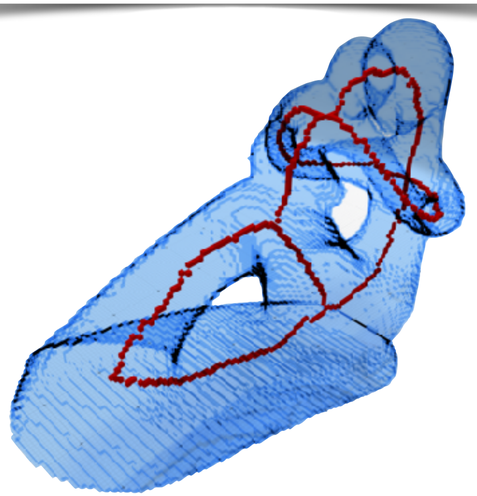
Volume = d-manifold in $\mathbb{R}^d$

k-manifold in $\mathbb{R}^d$, $k \leq d$

What Is topology preservation ?

Strongest

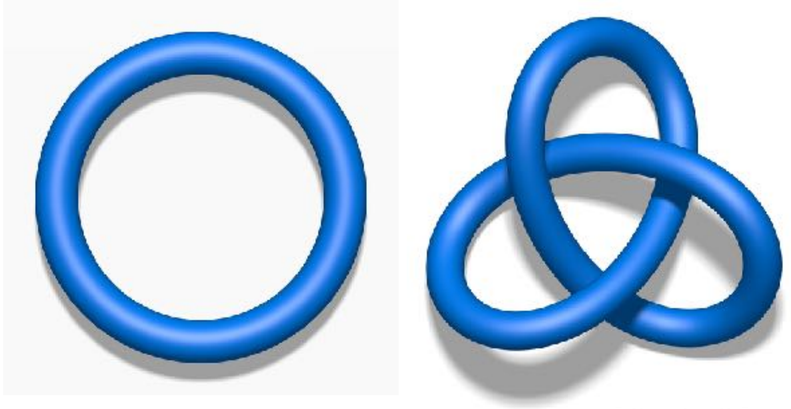
Homotopy equivalence



Homeomorphism

[RobertBrooks/Science Photo Library]

Ambient isotopy



[Jim.belk]

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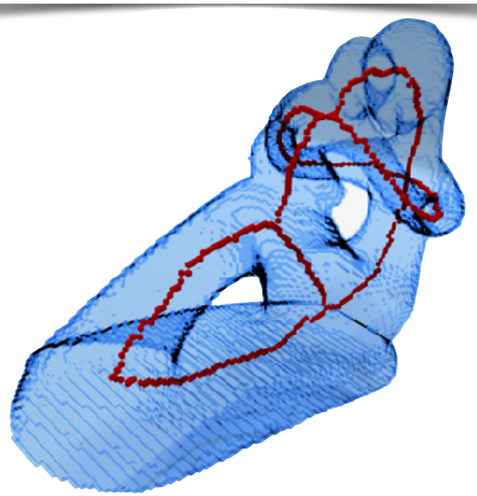

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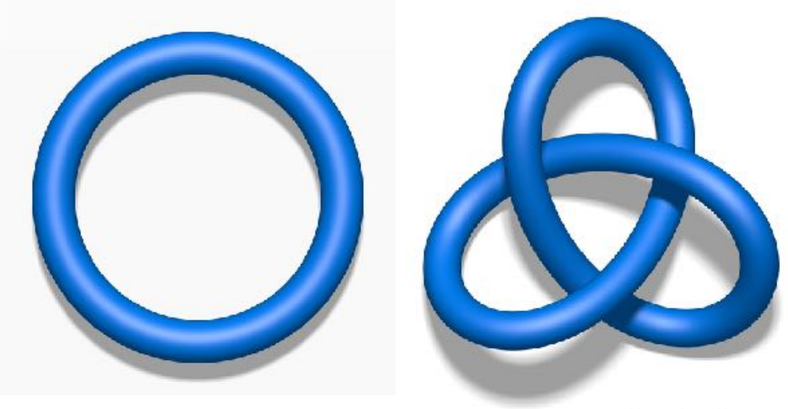

[Jim.belk]

Topology invariants

Strongest

# connected components	Genus Euler number	Homology groups	Cohomology rings	Fundamental group	Homotopy groups

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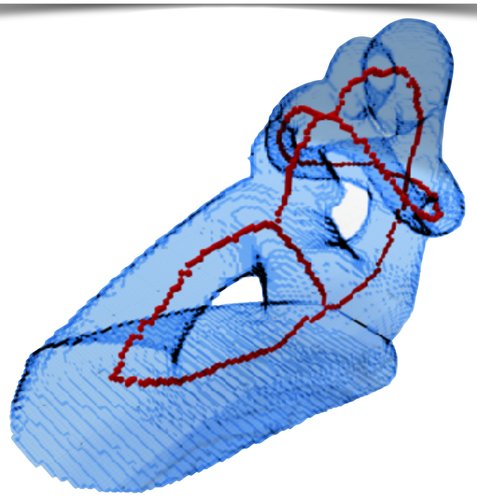

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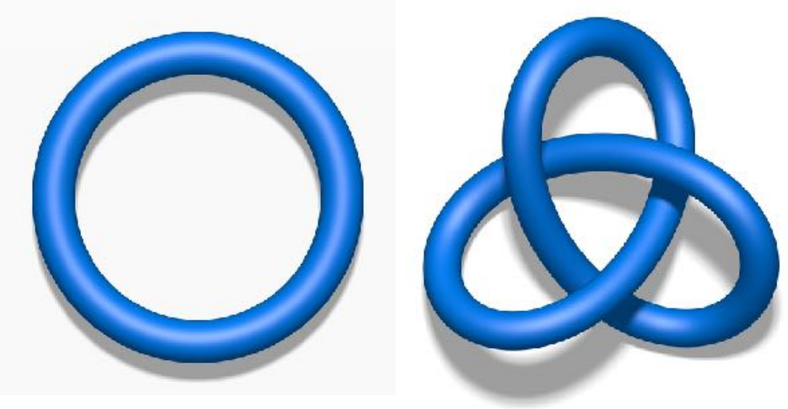

[Jim.belk]

Topology invariants

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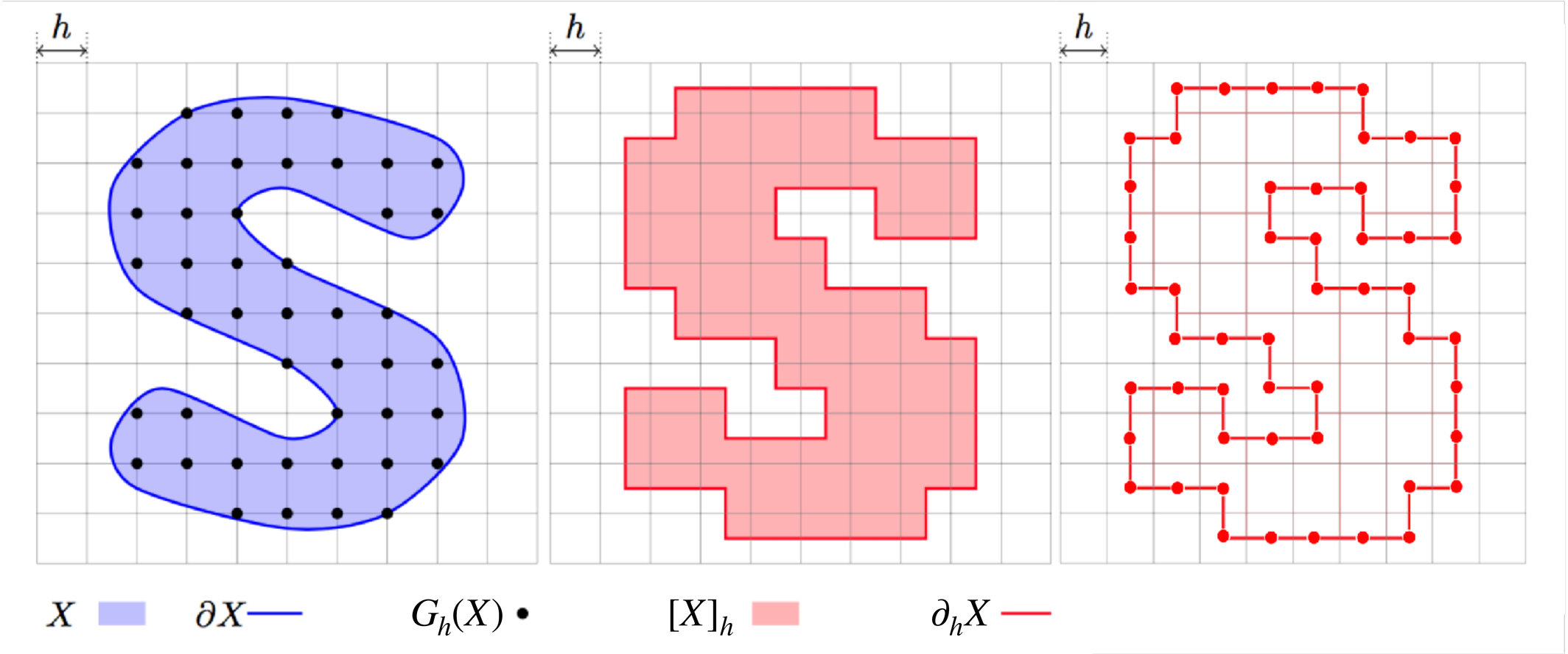
Computable

Decidable ?

Linking continuous and digital geometry : Gauss digitization with gridstep h



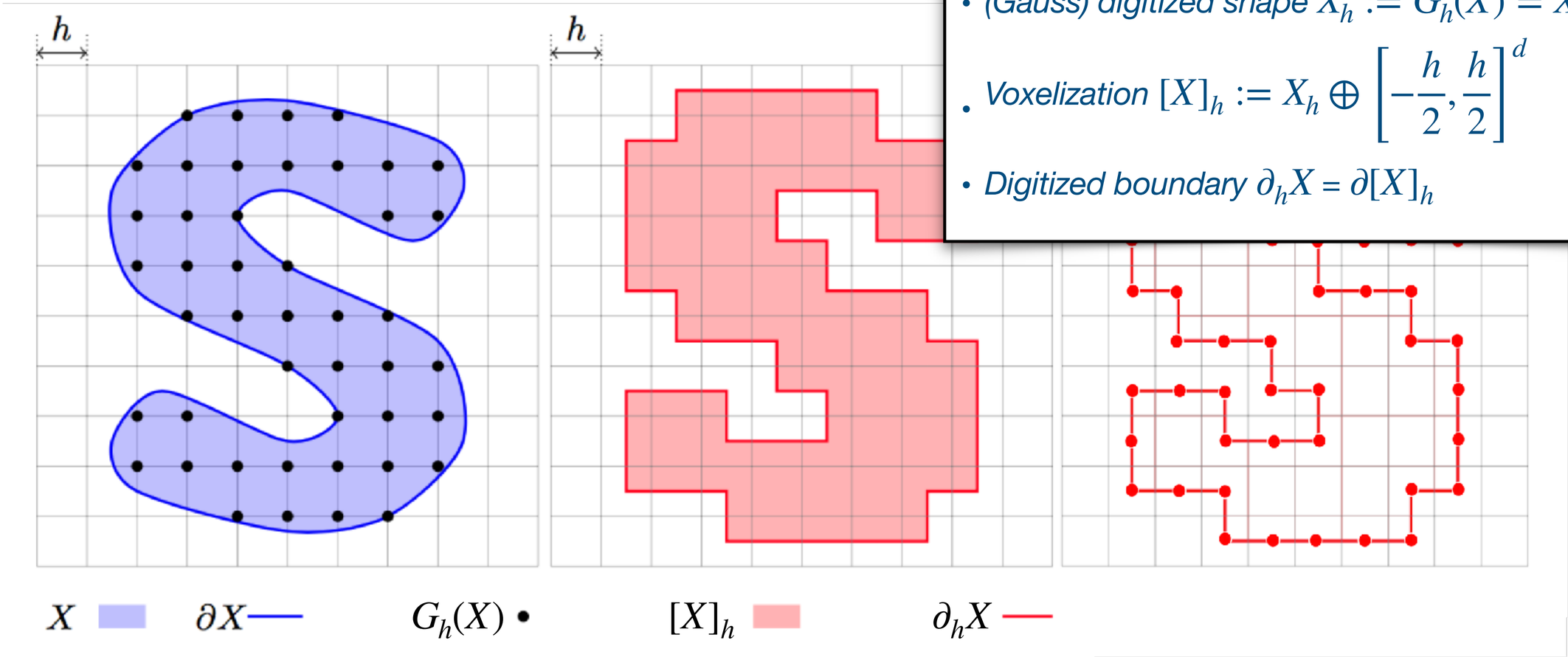
« digitization »

« voxelization »

« digitized surface »

Linking continuous and digital geometry : Gauss digitization with gridstep h

- *Euclidean shape* $X \subset \mathbb{R}^d$*:*
- *(Gauss) digitized shape* $X_h := G_h(X) = X \cap h\mathbb{Z}^d$
- *Voxelization* $[X]_h := X_h \oplus \left[-\frac{h}{2}, \frac{h}{2}\right]^d$
- *Digitized boundary* $\partial_h X = \partial [X]_h$



« digitization »

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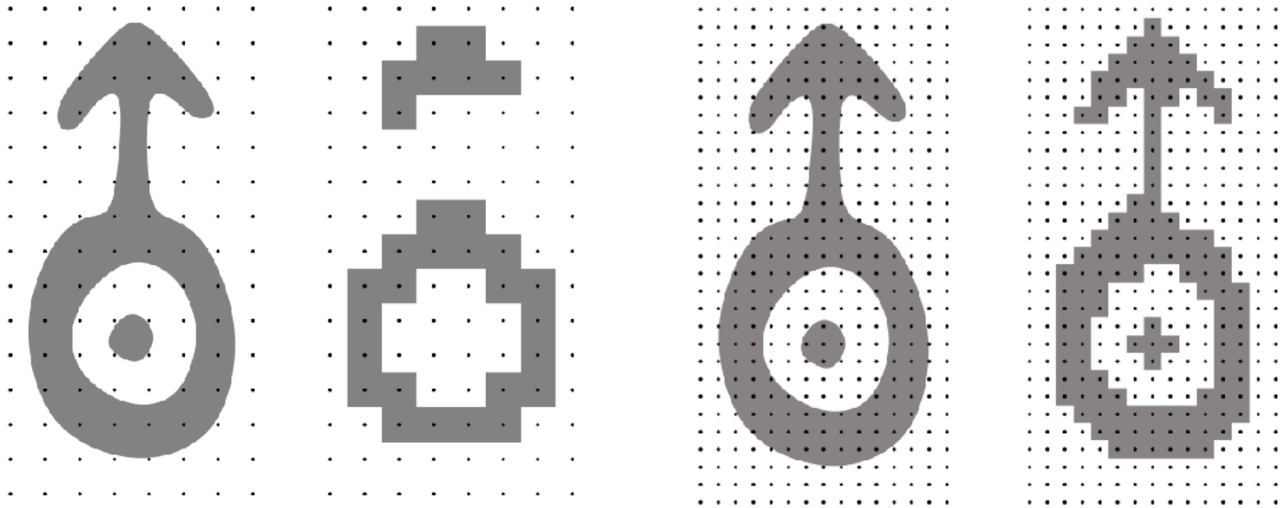
« digitized surface »

Homotopy equivalence through digitization



Theorem 11 (sampling theorem for n-dimensional binary images) *Let $r \in \mathbb{R}_+$ and $A \subset \mathbb{R}^n$ an r-regular set. Further let $\hat{A}$ be the reconstruction of A with an arbitrary r'-grid S in $\mathbb{R}^n$, $0 < r' < r$. Then A and $\hat{A}$ have identical homotopy trees and the Hausdorff distance between their boundaries is at most r'.*

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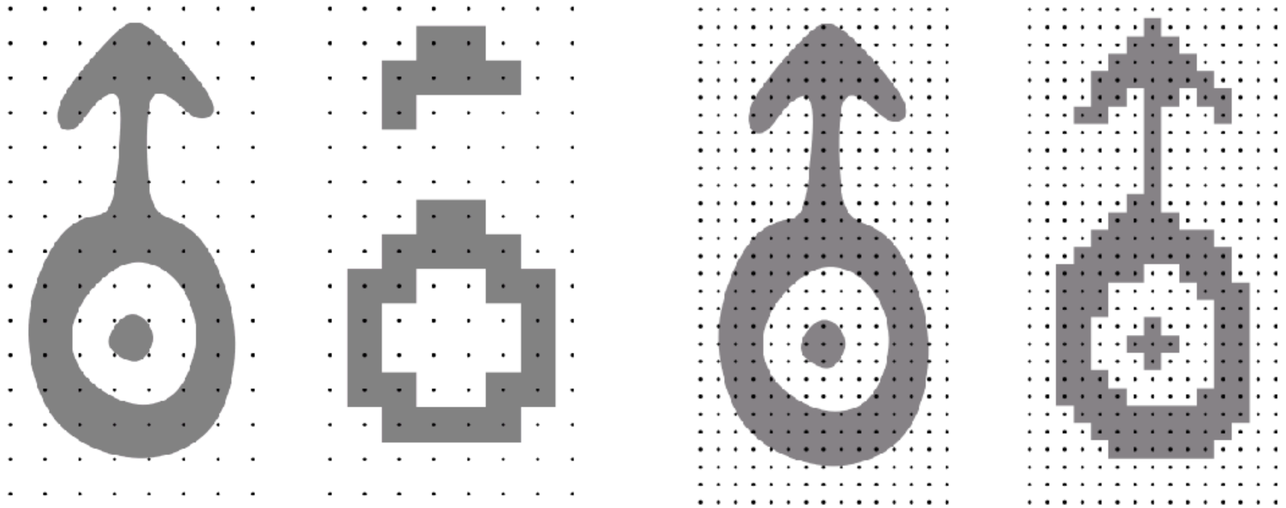

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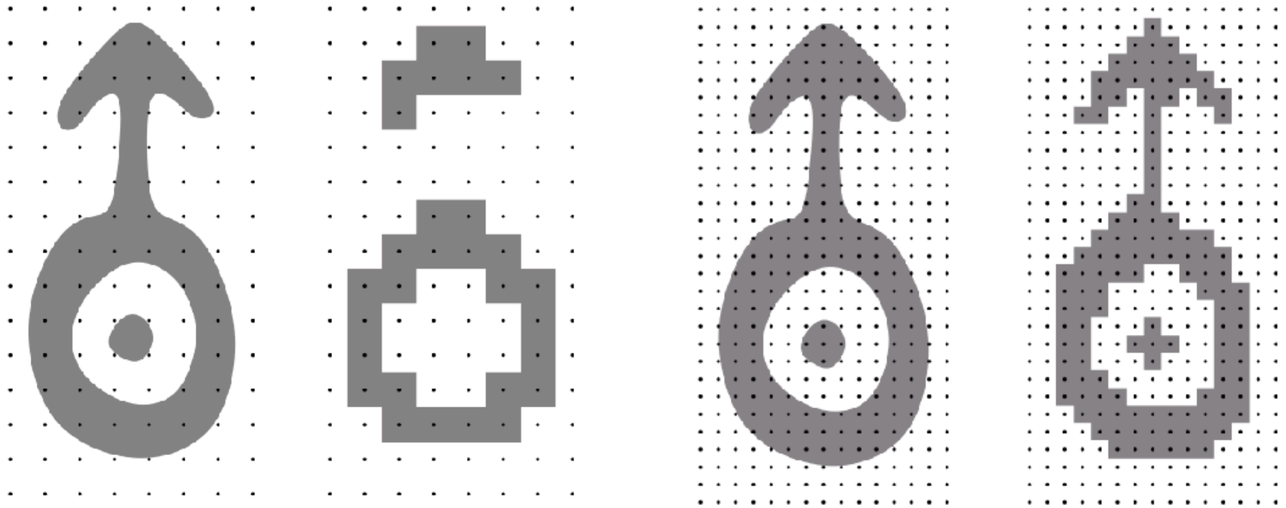

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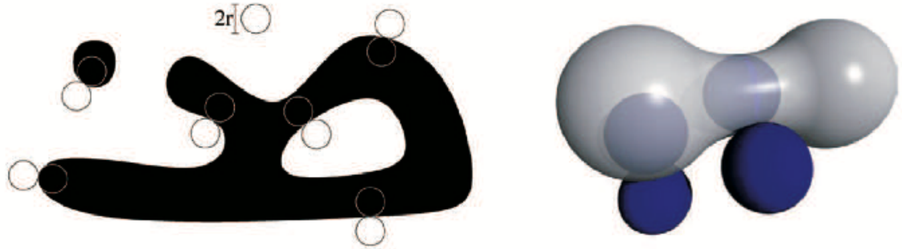
Class of shapes: r-regular shapes



(2D) (3D)

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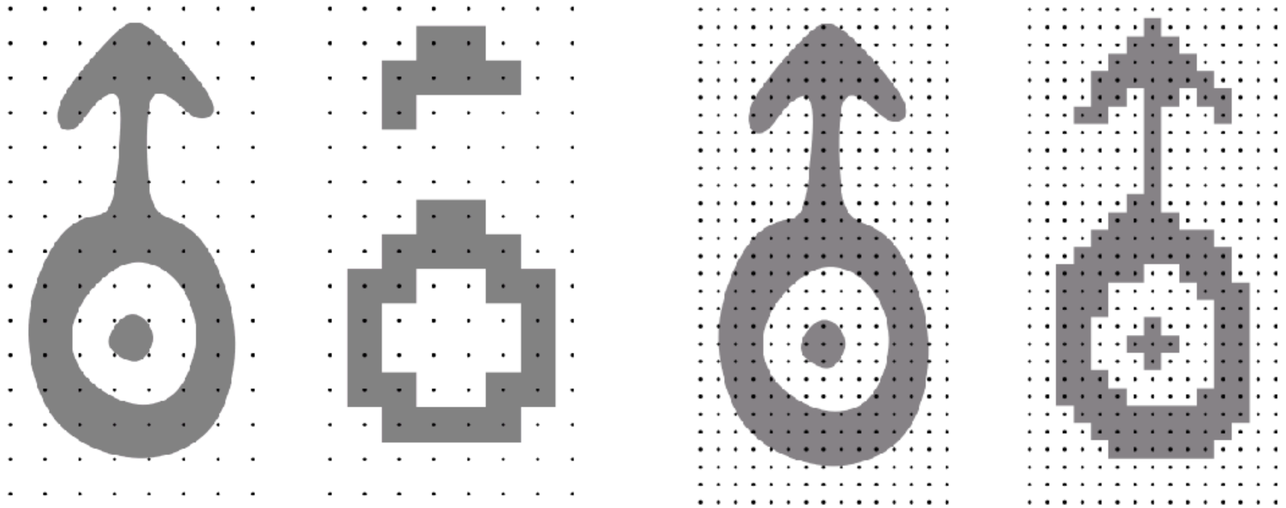

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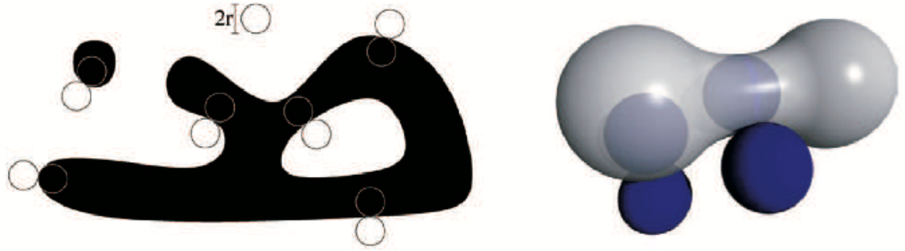

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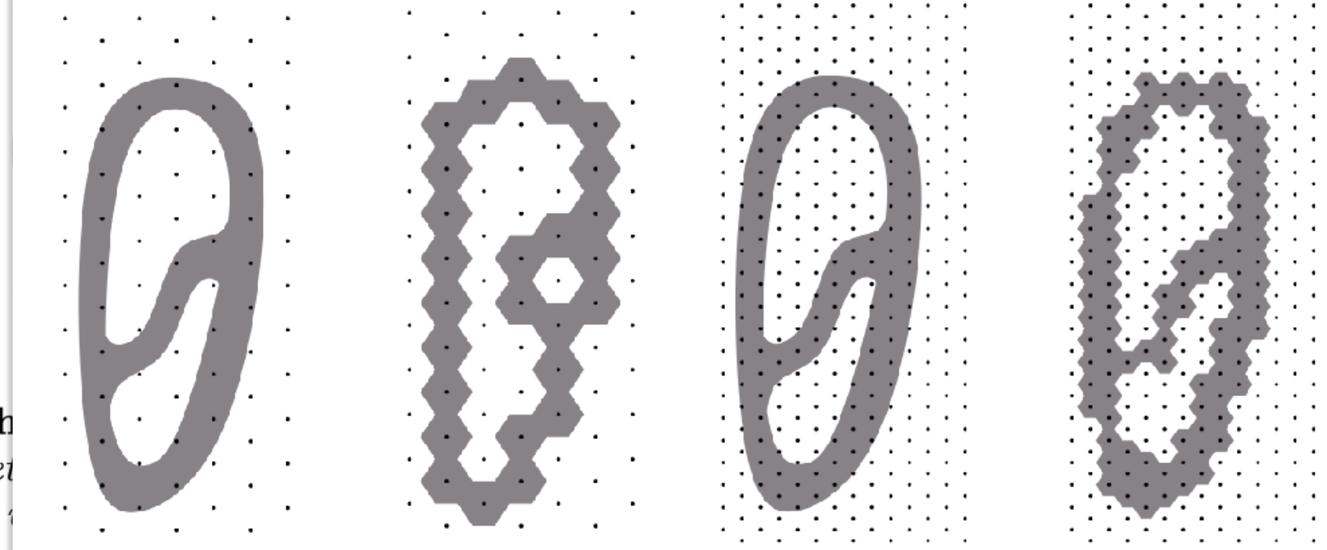
$\forall h < r$, A and $[A]_h$ are homotopy equiv.
$\mathbb{R}^d \backslash A$ and $\mathbb{R}^d \backslash [A]_h$ are homotopy equiv.

Homotopy equivalence through digitization

Works also for hexagon sampling



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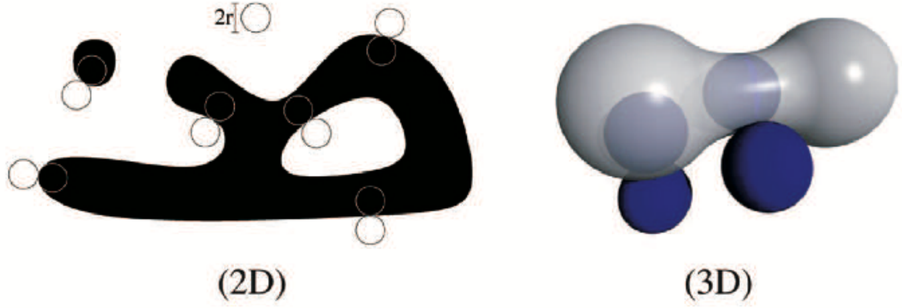

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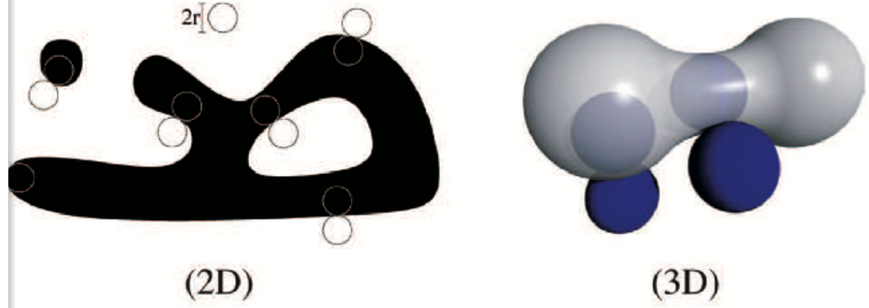
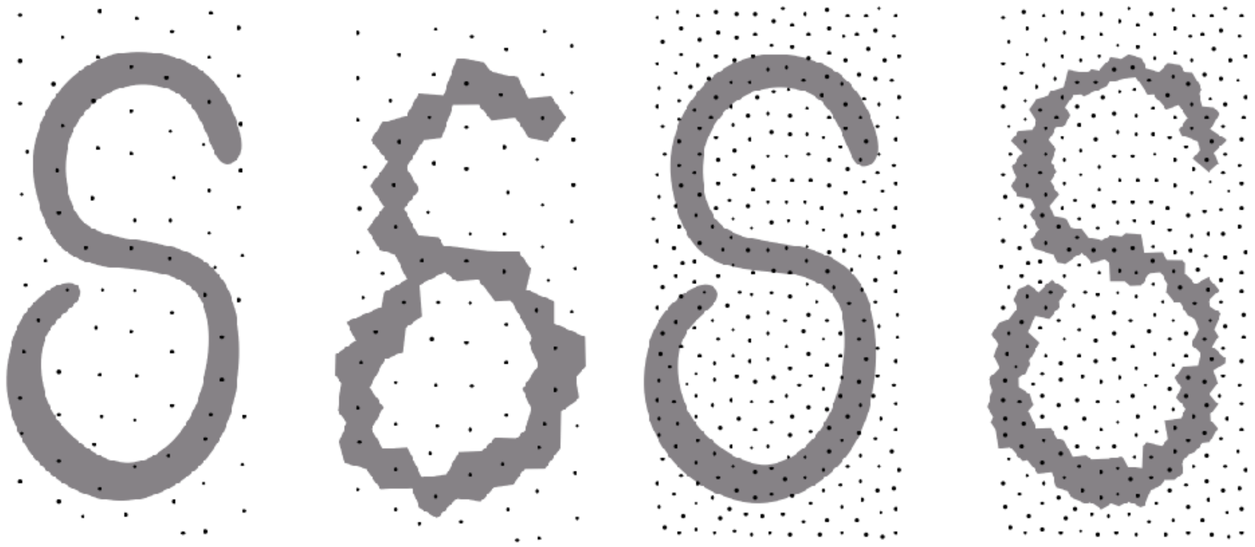


In fact for any dense sampling !



Th
Let
A
hor
r'.

[Chazal, Lieutier 2005]

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Ambient isotopy in 2D

(2D only)

Let A be a r-regular shape in $\mathbb{R}^2$

$\forall h < r$, A and $[A]_h$ are ambient isotopic

[Stelldinger, Köthe 05]

Ambient isotopy in 2D … only

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Let A be a r-regular shape in $\mathbb{R}^2$

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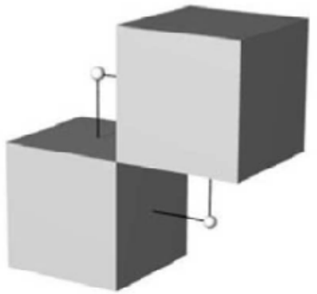
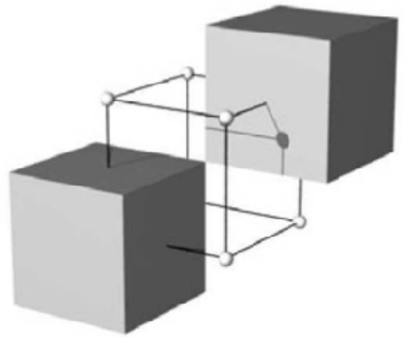
[Stelldinger, Köthe 05]

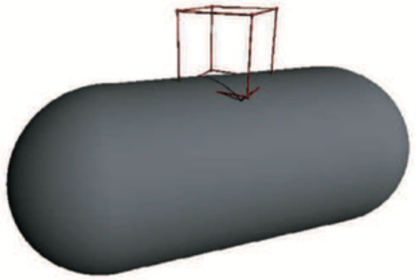
non-manifold.



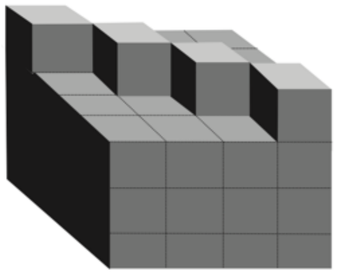


In 3D, some r-regular shapes

are not digitized as manifolds, whatever h



Dig.



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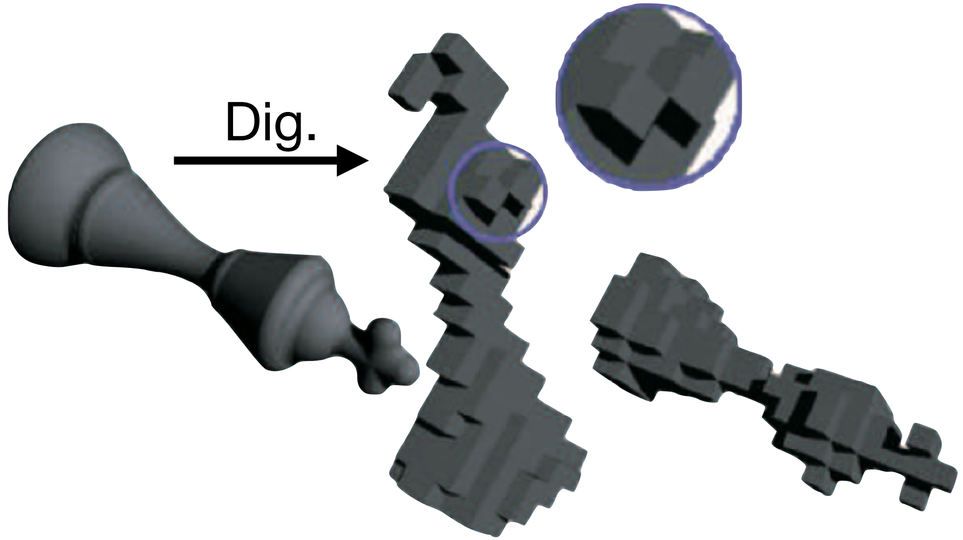
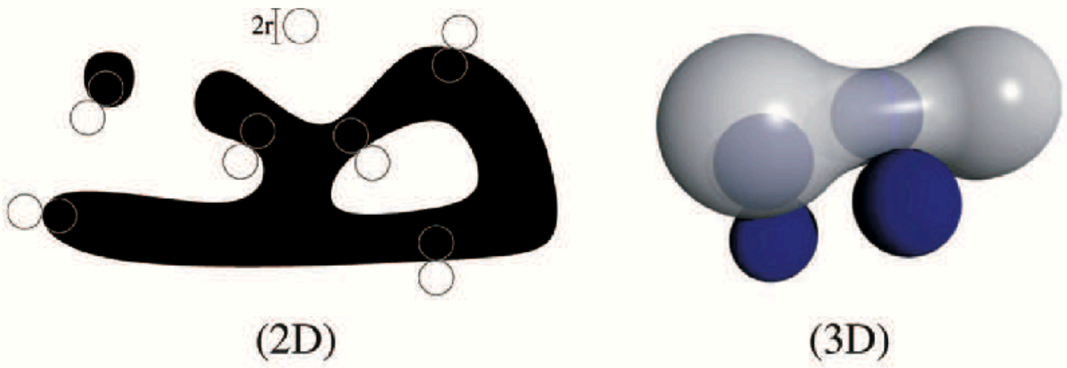


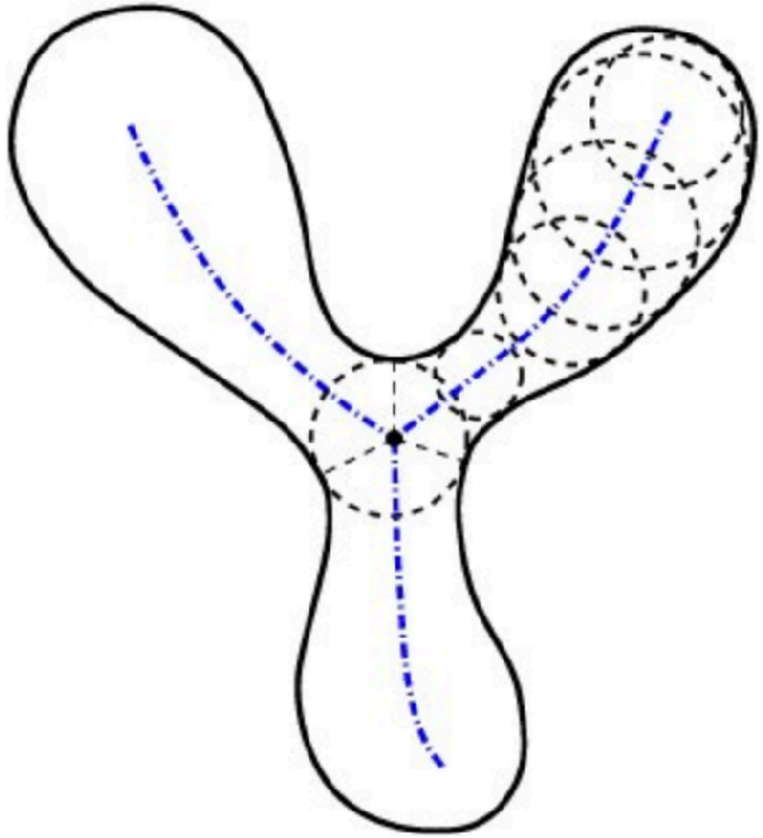
(Historical note)

Class of shapes: r-regular shapes



r**-regularity** of $X \subset \mathbb{R}^d$
Inside and outside osculating
r-balls everywhere on ∂X

[Pavlidis82, Serra 82]



Euclidean shape $Y \subset \mathbb{R}^d$***:***

- *Medial axis M(Y) := set of points that have at least 2 closest points on Y*
- ***Reach(Y)*** *= infimum of distances between Y and M(Y)*

[Federerer 59]

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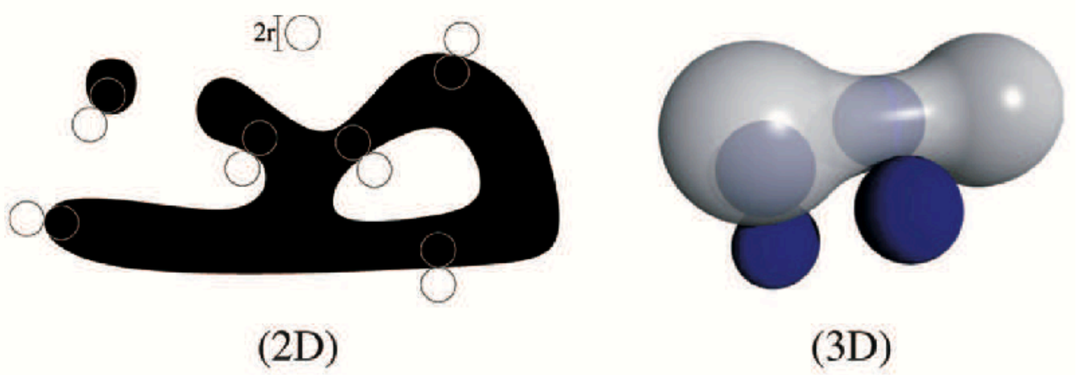
Class of shapes: r-regular shapes



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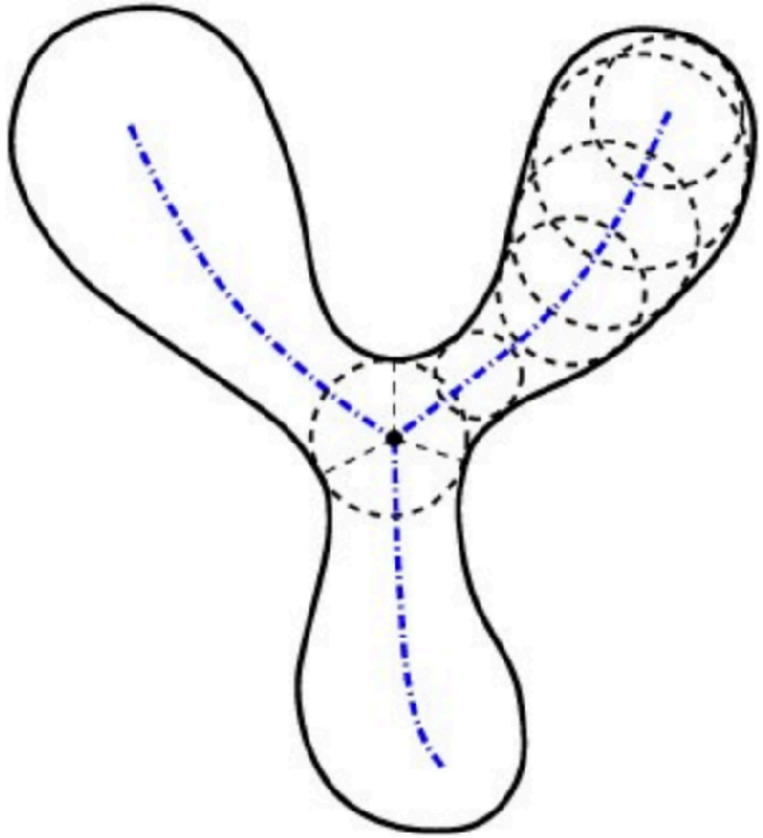

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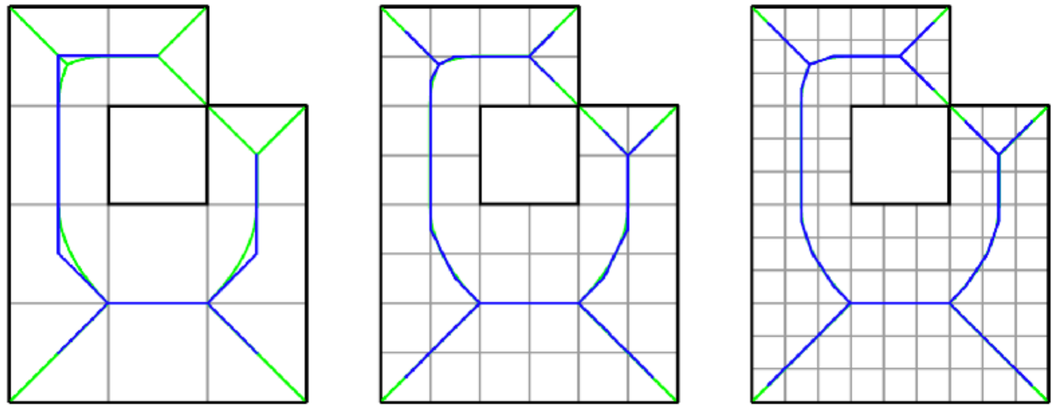
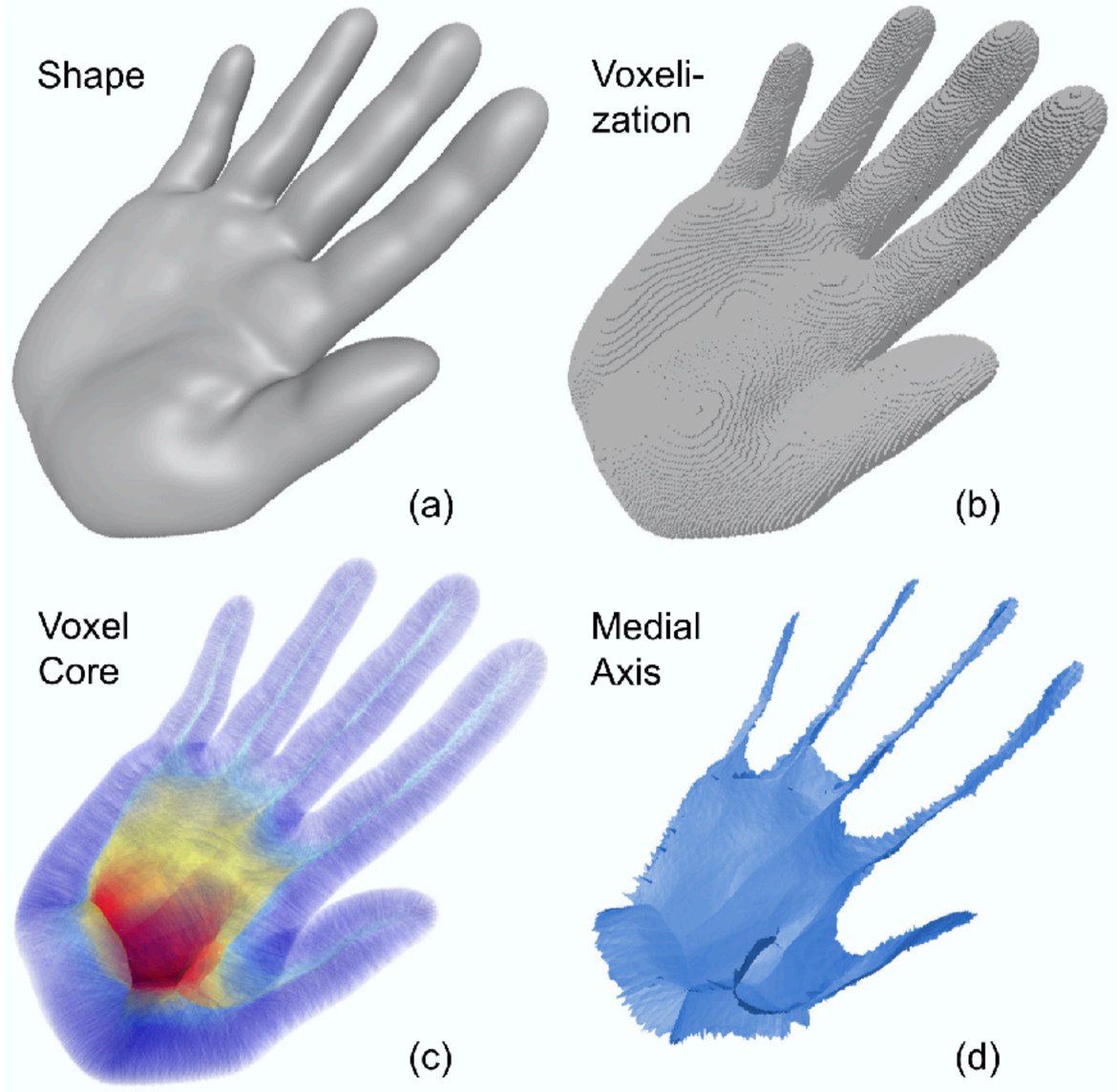
r-**regularity** of $X \subset \mathbb{R}^d \Leftrightarrow \mathrm{reach}(\partial X) \geq r$ [Federer 59]

Medial axis homotopy equivalence

For $X \subset \mathbb{R}^d$ with $\rho := \text{reach}(\partial X) > 0$,
X and $[X]_h$ and its voxel core are homotopy equivalent, for $h < 2\rho/\sqrt{3}$.

[Yan, Letscher, Ju 18]





Some recent works on topology inference

I. More generic class of shapes

- Quasi-r-regularity (2d)
  [Ngo, Passat, Kenmochi, Debled-Rennesson 19]
- Locally turned-bounded curves (2d)
  [Le Quentrec, Mazo, Baudrier, Tajine 20]
- Locally turned-bounded surfaces (3d)
  [Macke, Baudrier, Le Quentrec 25]

Includes polygons with obtuse angles

Quasi r-regularity and (θ, δ)-**LTB** for $0 < \theta \leq \pi/2$ implies ambient isotopy for small enough grid step

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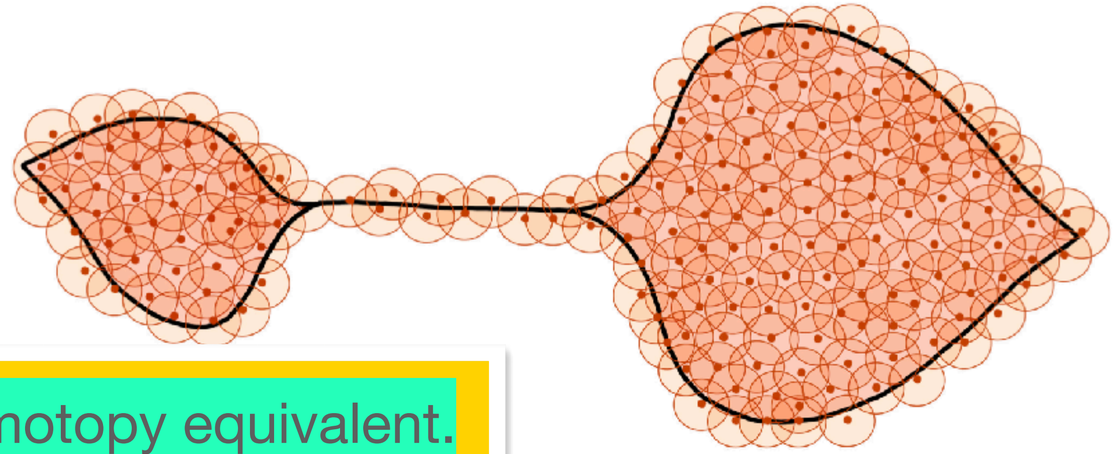
II. Sample points in $\mathbb{R}^d$

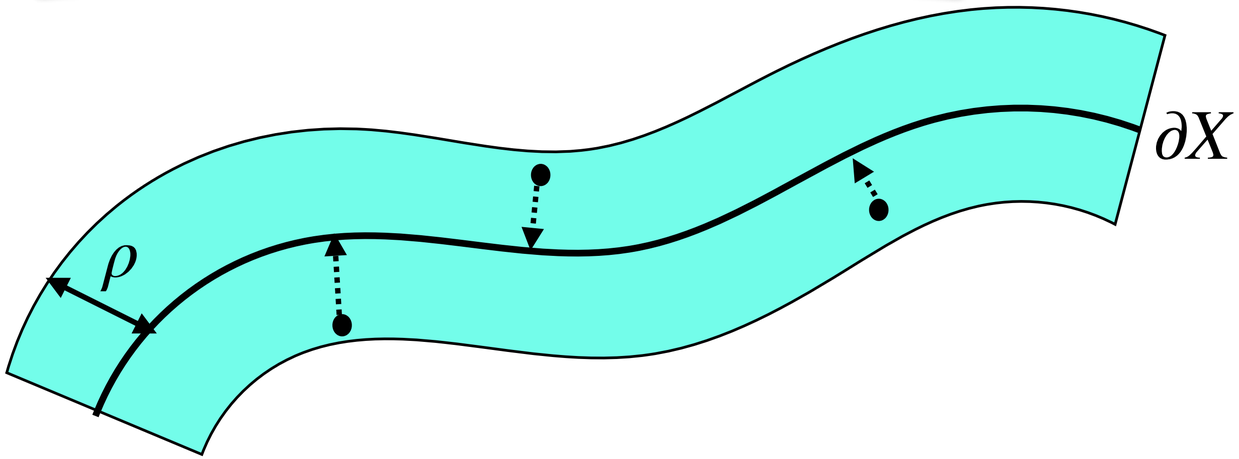
- Union of balls $P \oplus B_r$ covering X
  [Attali, Kouřimská, Fillmore, Ghosh, Lieutier, Stephenson, Wintraecken 22]



Under some hyp., $\exists r$, X and $P \oplus B_r$ are homotopy equivalent.

Projection on smooth boundary $\pi_{\partial X}$

Any point within the reach ρ
has a unique $\perp$ projection on ∂X



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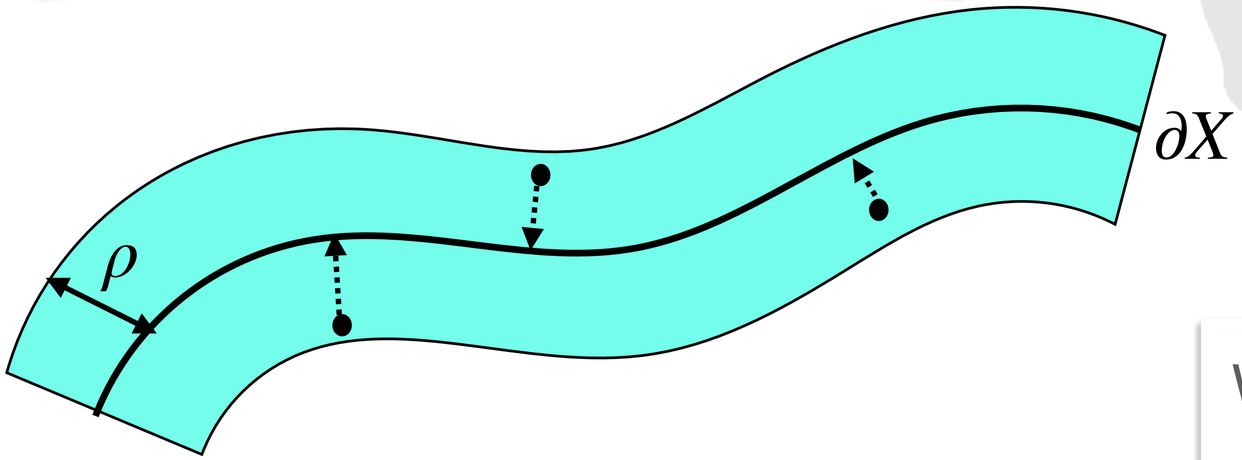

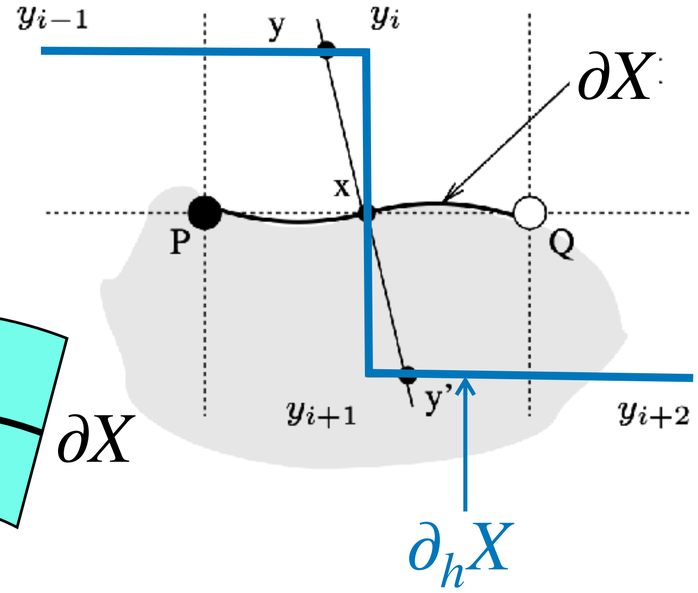

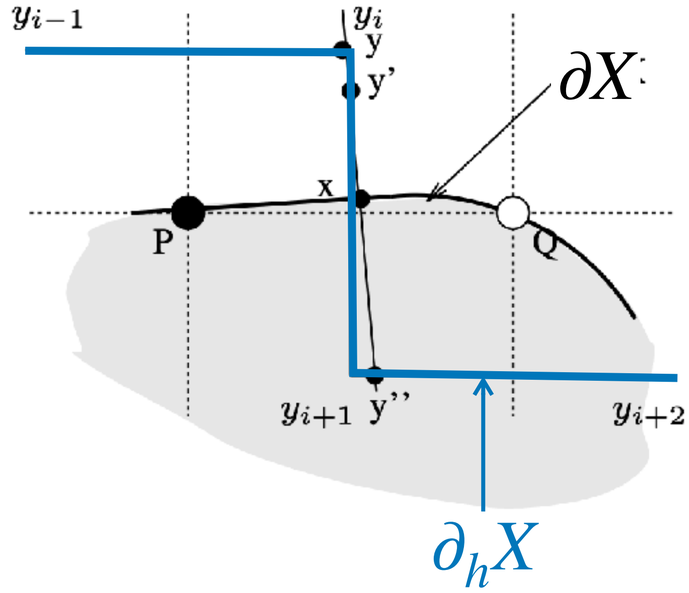

Whatever the grid step, multiple points of $\partial_h X$
may project onto the same point of ∂X

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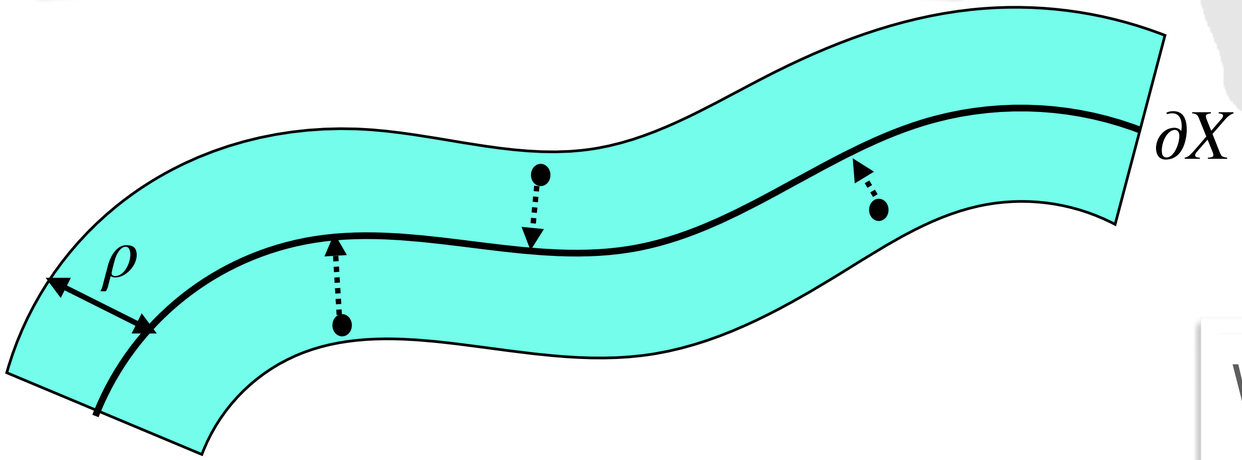
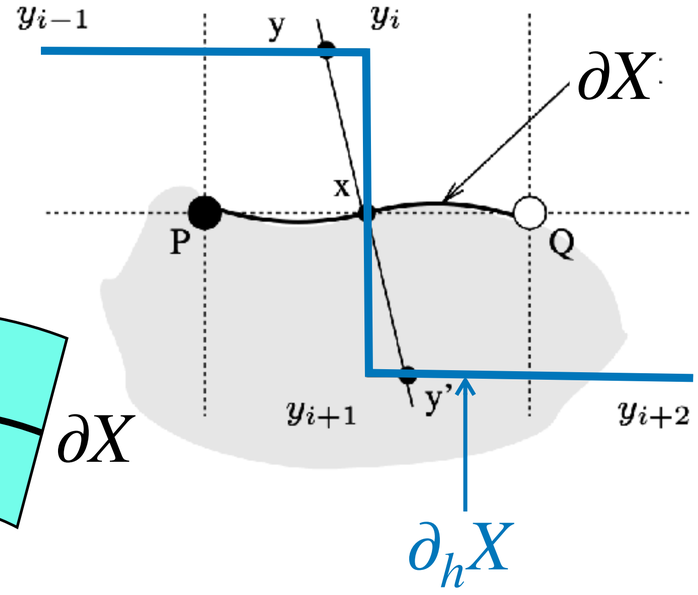
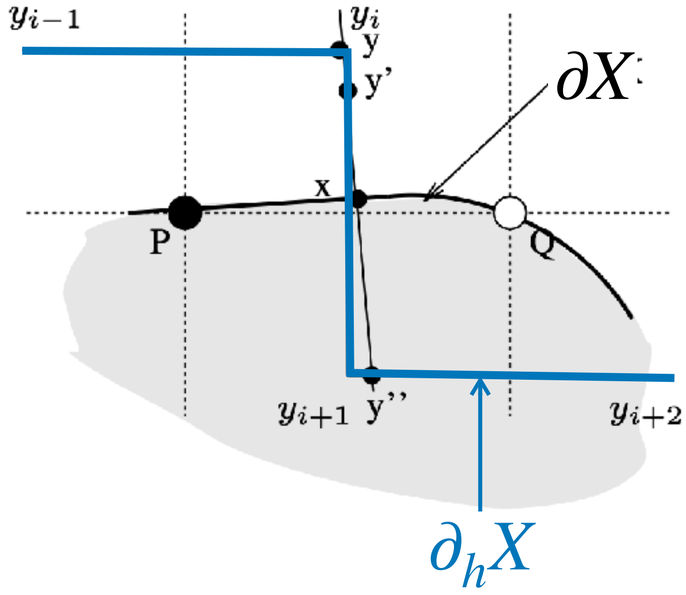


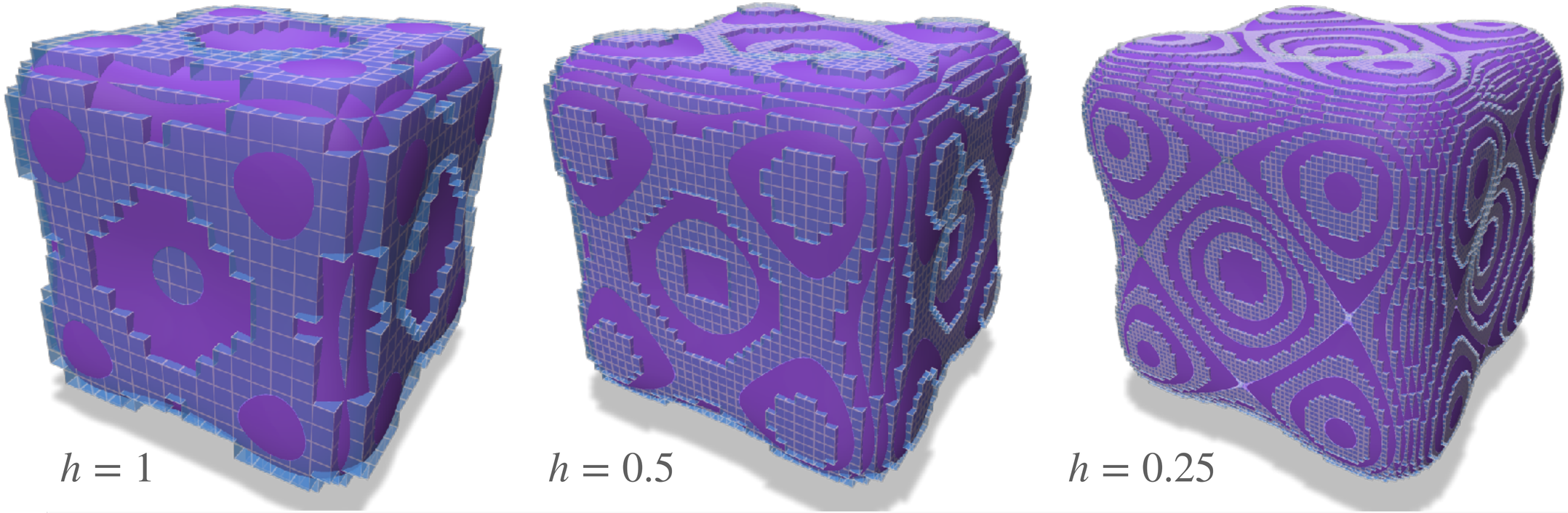




Whatever the grid step, multiple points of $\partial_h X$
may project onto the same point of ∂X

There exists $X \subset \mathbb{R}^d$, compact and even convex sets with $\mathrm{reach}(\partial X) > 0$
such that the projection $\pi_{\partial X} : \partial_h X \to \partial X$ **is not injective.**

Boundaries ∂X and $\partial_h X$ are close

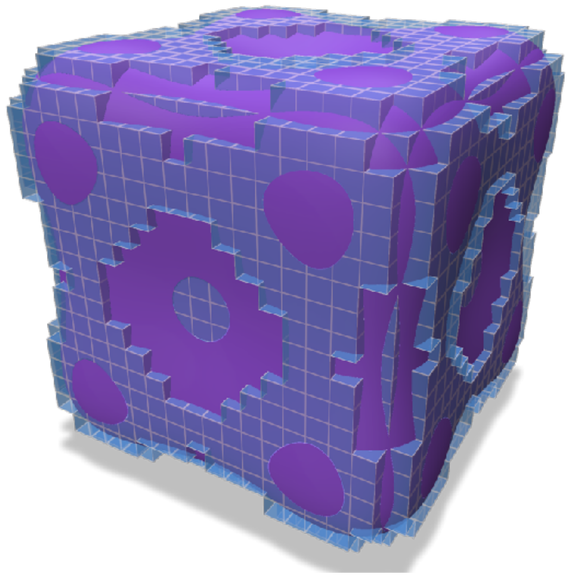


For $X \subset \mathbb{R}^d$ any compact domain with $\rho := \text{reach}(\partial X) > 0$, then

$$d_H(\partial X, \partial_h X) \leq \frac{\sqrt{d}}{2} h \text{ for } h < \frac{2\rho}{\sqrt{d}}.$$ [Stelldinger, Köthe 2005]

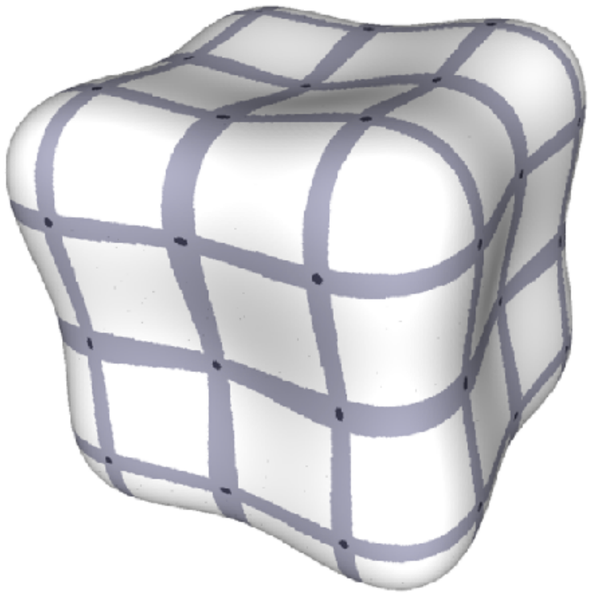
Bound is tight, closest point through projection $\pi_{\partial X}$

Bijectivity of projection and manifoldness

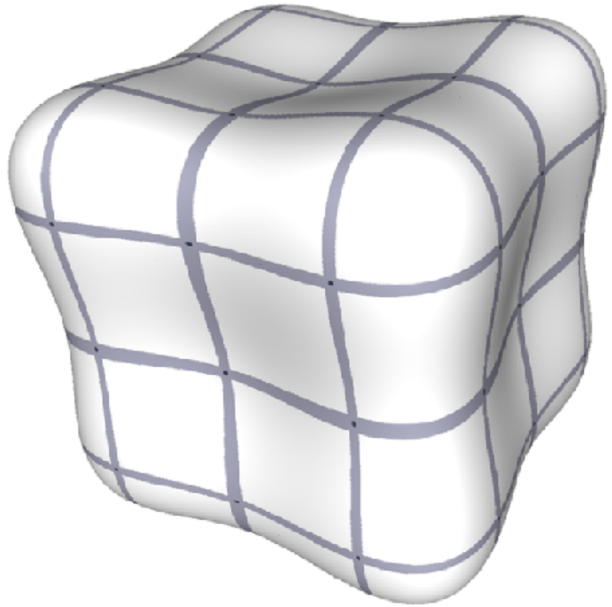


∂X has positive reach

$h = 0.1$



$h = 0.05$



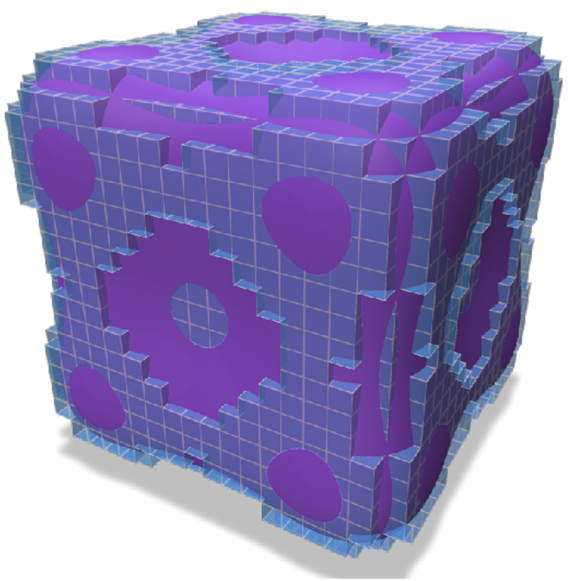
$h = 0.025$

Area of non-injective part of $\pi_{\partial X} : \partial X_h \to \partial X$ tends to zero as $h \to 0$.
(light gray + dark gray zones $\approx O(h)$)

Area of non-manifoldness part of ∂X_h tends quickly to zero as $h \to 0$.
(dark gray zones $\approx O(h^2)$)

[L, Thibert 16]

Bijectivity of projection and manifoldness



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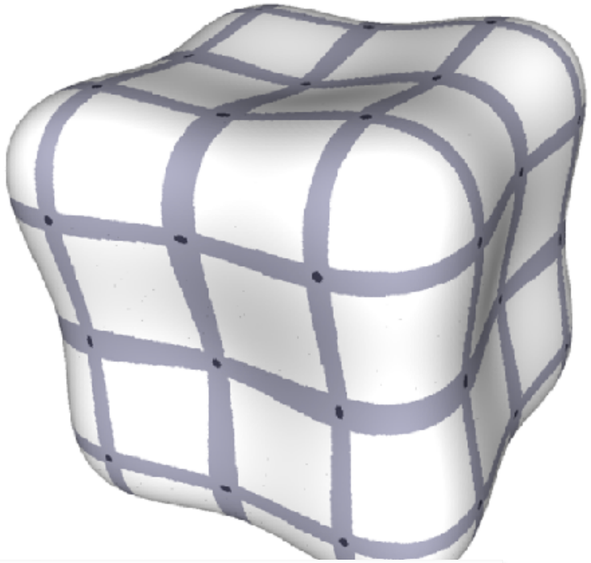

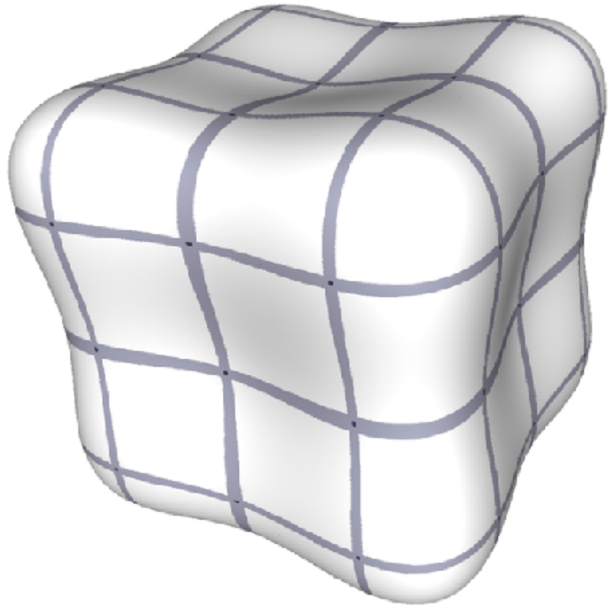

$h = 0.025$

We even know **where** problem occurs !
(near axes or orthant changes)

Area of non-injective part of
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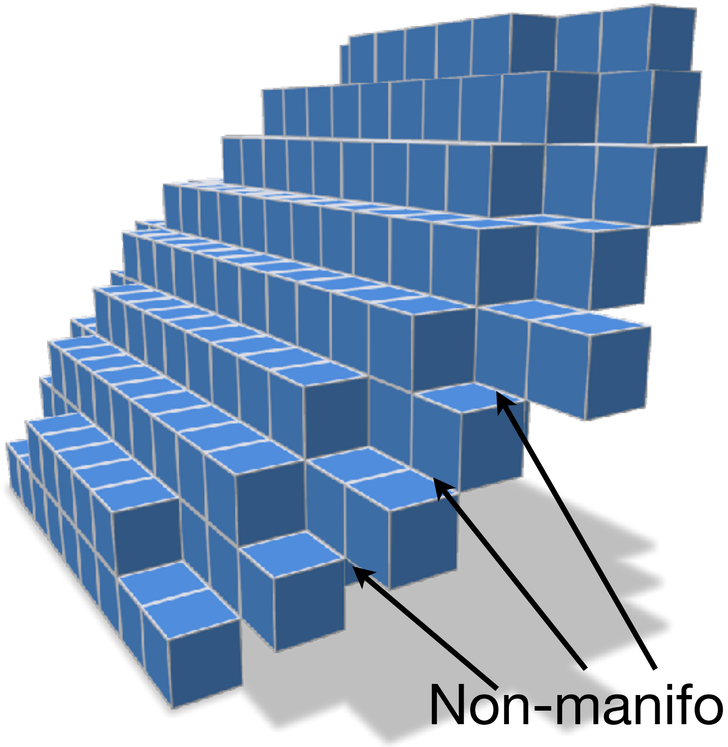
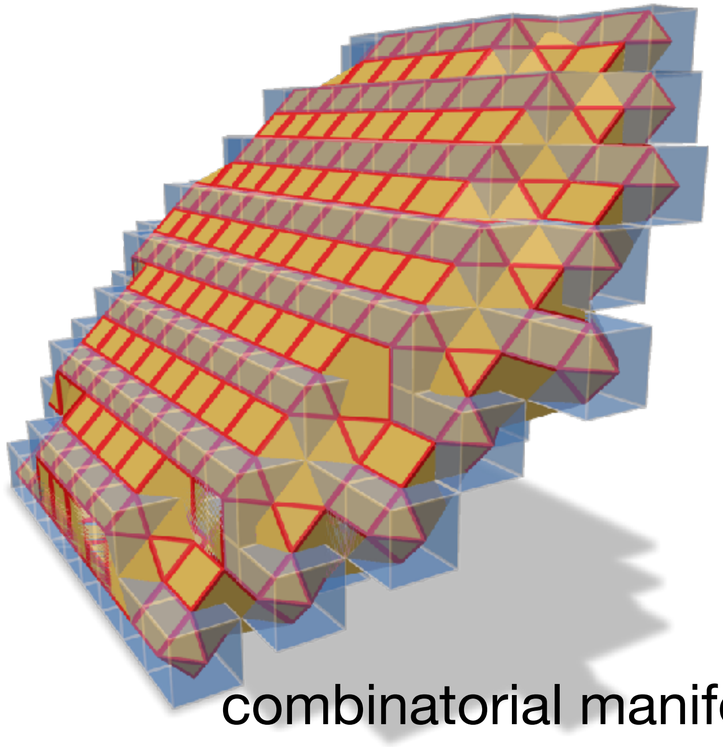
Digital surfaces + topology (primal ↔ dual)



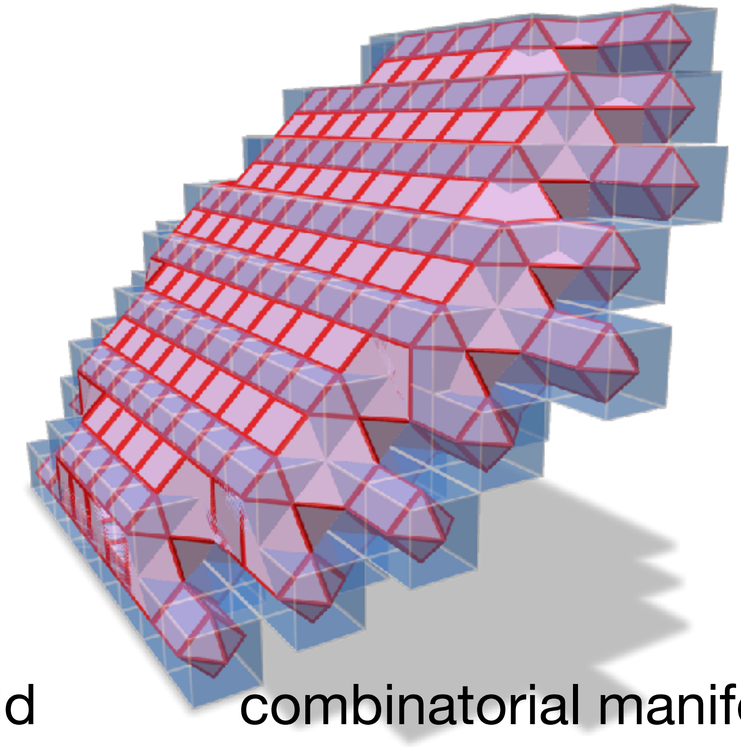
Non-manifold

Primal surface
(digitized ellipsoid)



combinatorial manifold

Dual surface
(26,6) topology



combinatorial manifold

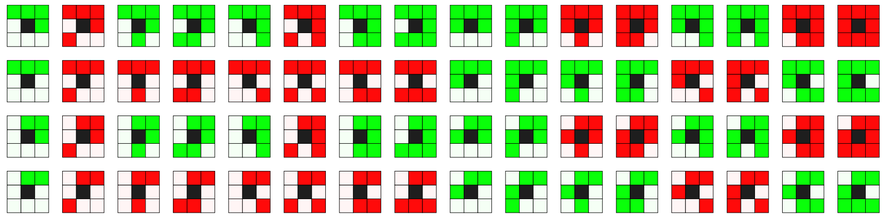
Dual surface
(6,26) topology

Adding object/background topology allows **manifoldness** in arbitrary dimensions

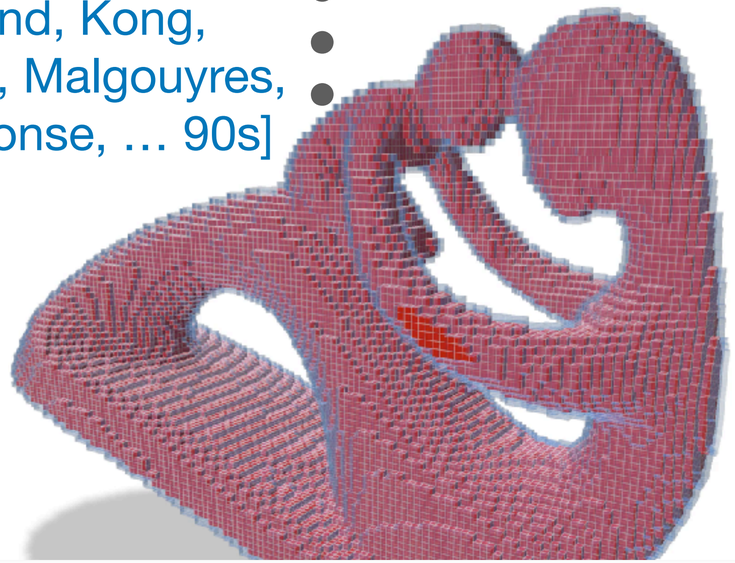
- exactly $d-1$ paths crossing at each point
- Jordan-Brouwer separation theorem [L, Montanvert 00]

Homotopy invariant deformations

Simple points: points whose removal or addition do not change homotopy



[Bertrand, Kong, Malandain, Malgouyres, Fourey, Ronse, … 90s]



Very fast: look-up tables in 2d and 3d
Homotopic skeletons

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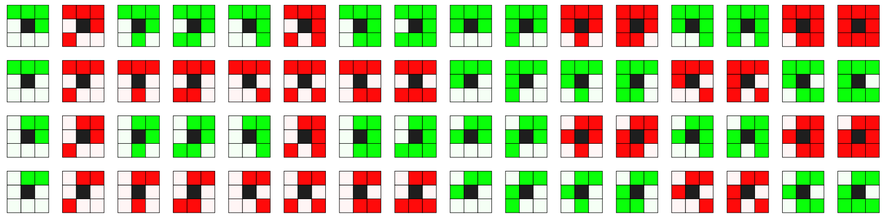

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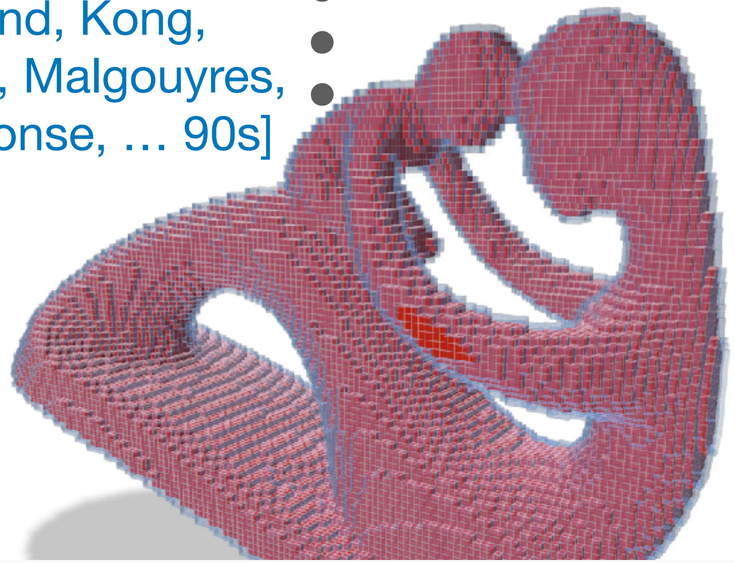

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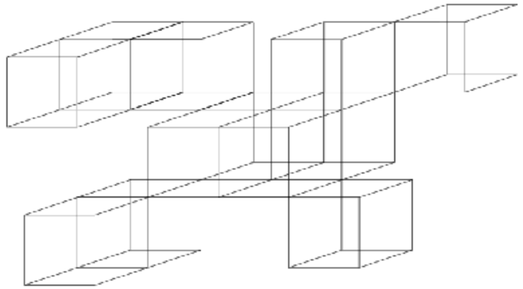
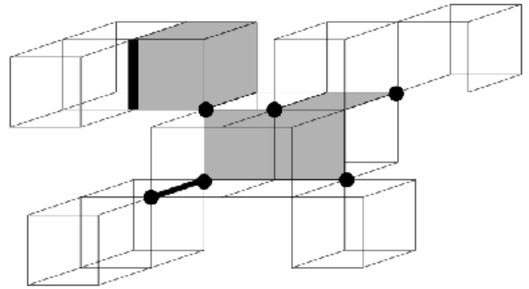
[Bertrand, Couprie 2008]





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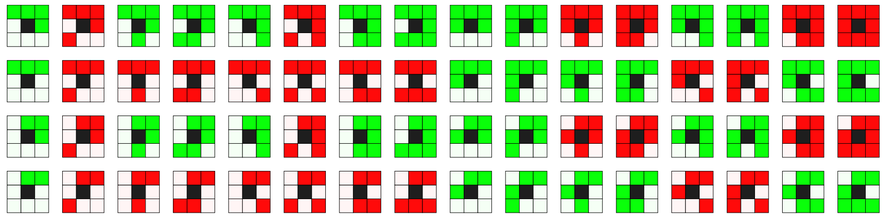

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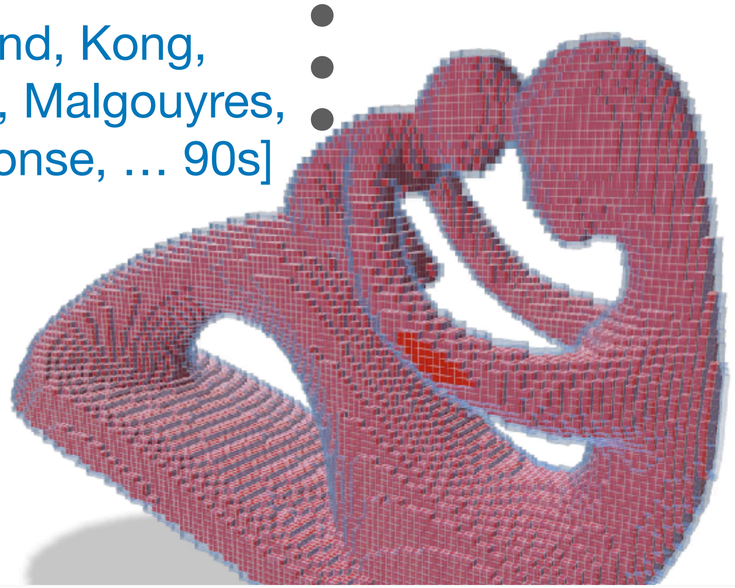
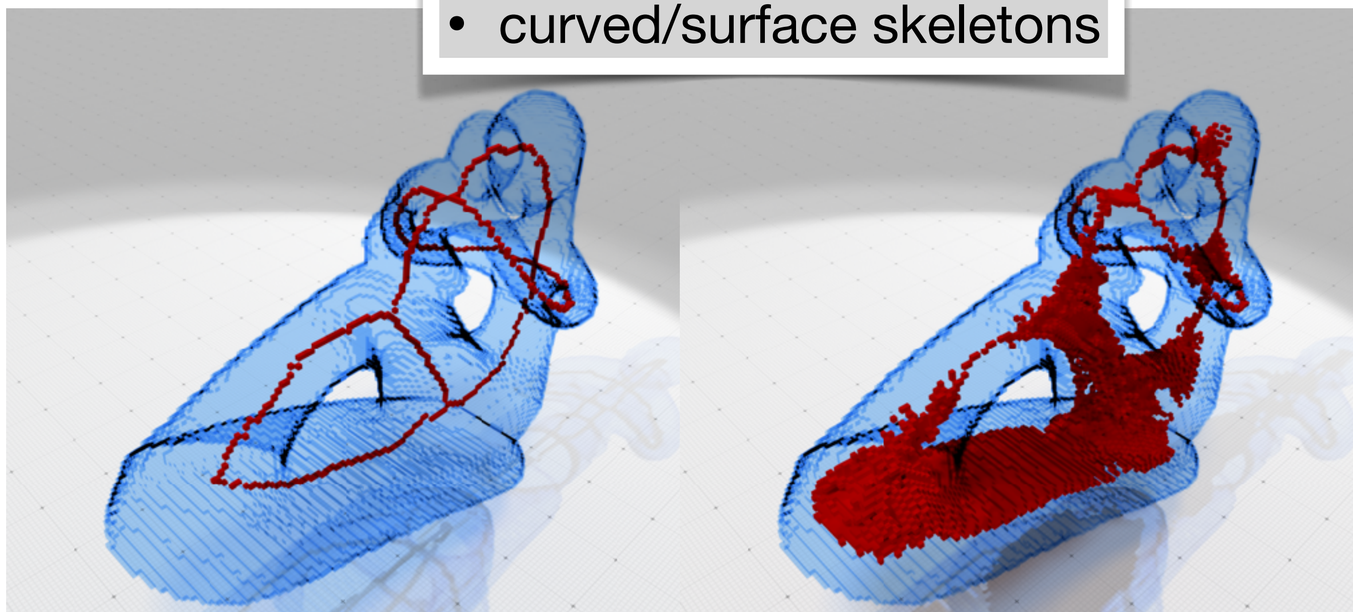


Very fast: look-up tables in 2d and 3d
Homotopic skeletons

Critical cells: cells that do not collapse onto their neighborhood

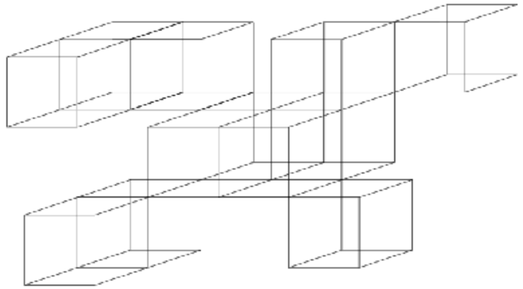
- parallel computations
- curved/surface skeletons

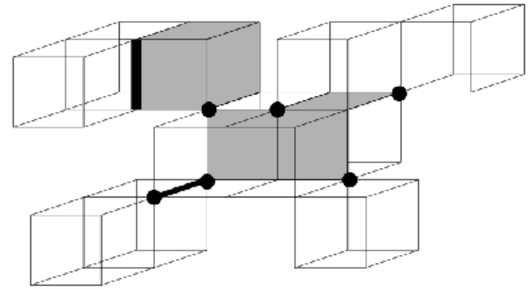


[Chaussard, Couprie, Kuba, Palágyi, … 25]

Homotopy invariant deformations

[Bertrand, Couprie 2008]





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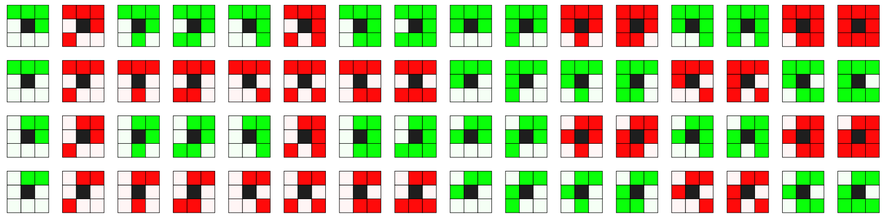
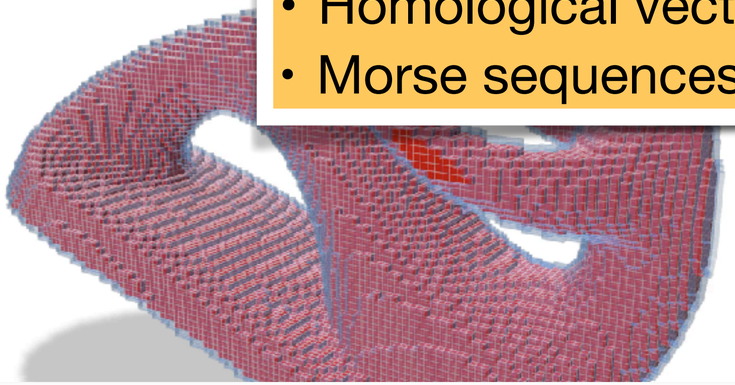


Critical cells: cells that do not collapse onto their neighborhood

[Bertrand, Kong, Malandain, Malgouyres Fourey, Ronse, … 90s]

- Related to **Topological Data Analysis**
- Many recent developments in persistent homology, Morse theory
- Homological vector fields [Gonzalez-Lorenzo, Bac, Mari, Real 17]
- Morse sequences [Bertrand 24] [Bertrand, Najman 25]



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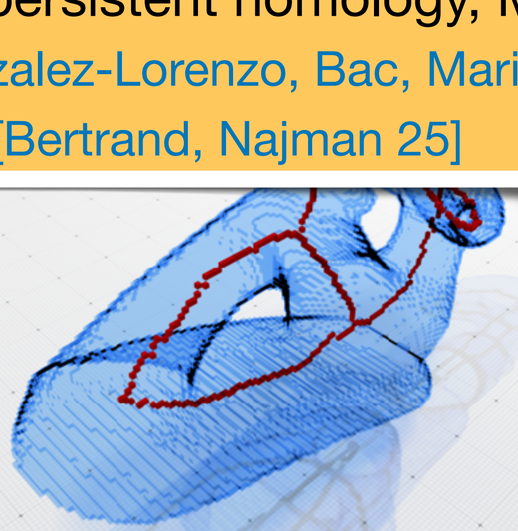
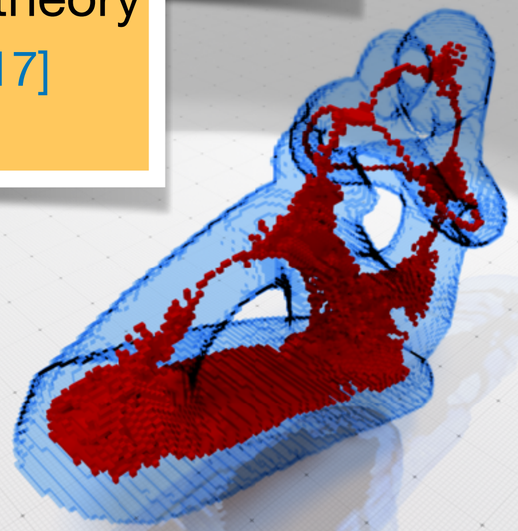
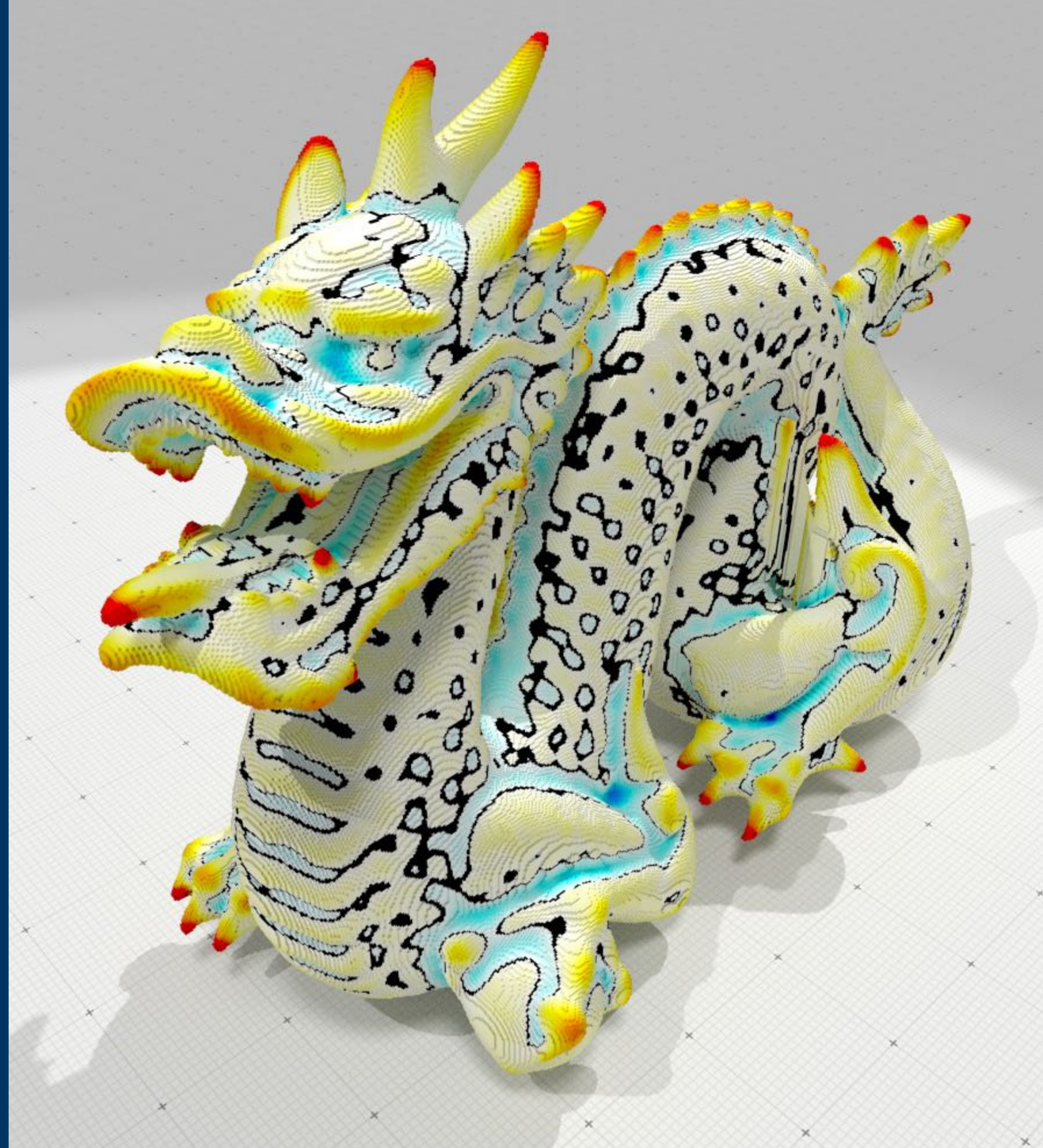
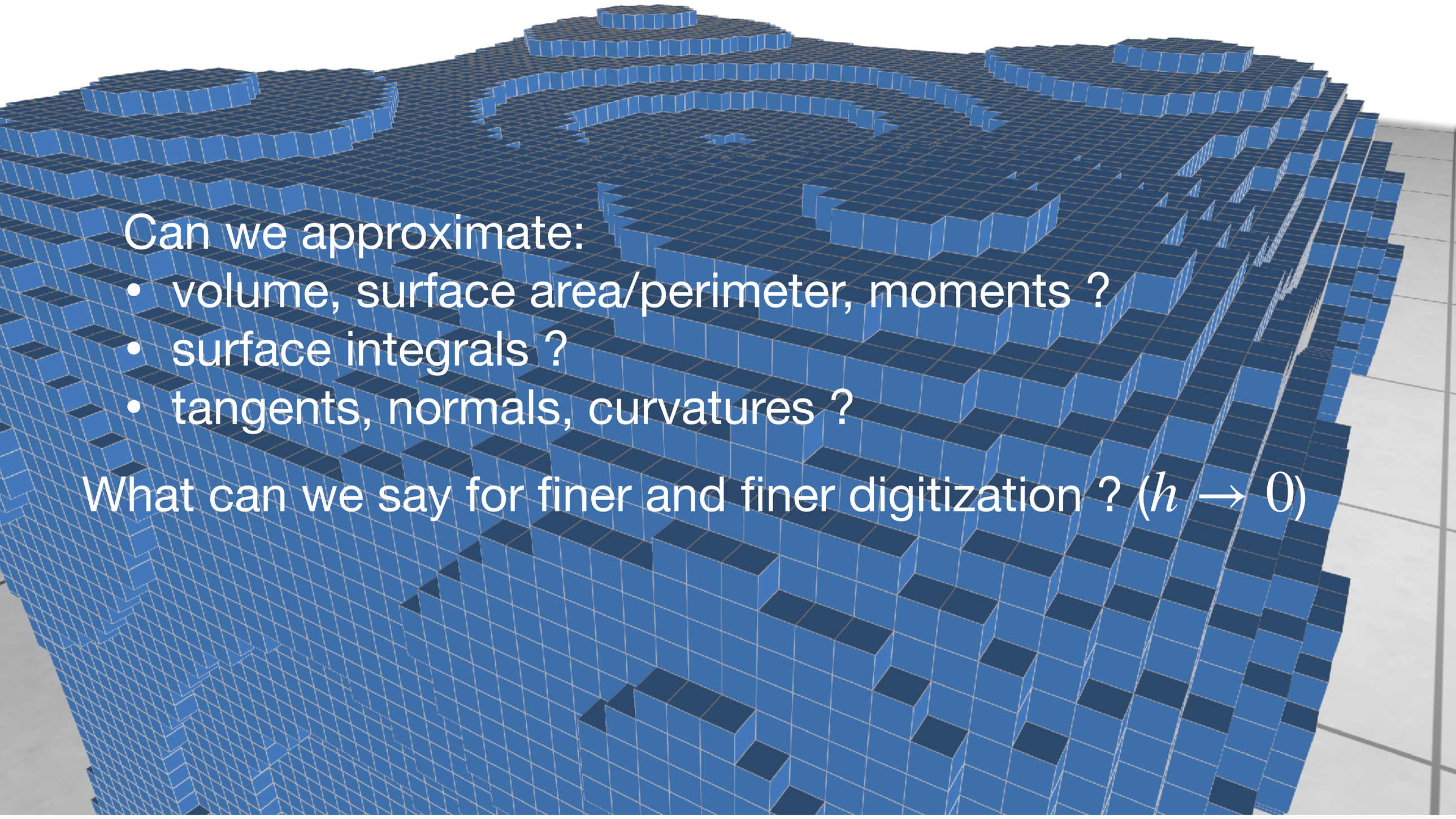




[Chaussard, Couprie, Kuba, Palágyi, … 25]

Geometric inference and multigrid convergence




Can we approximate:
• volume, surface area/perimeter, moments ?
• surface integrals ?
• tangents, normals, curvatures ?
What can we say for finer and finer digitization ? ($h \to 0$)

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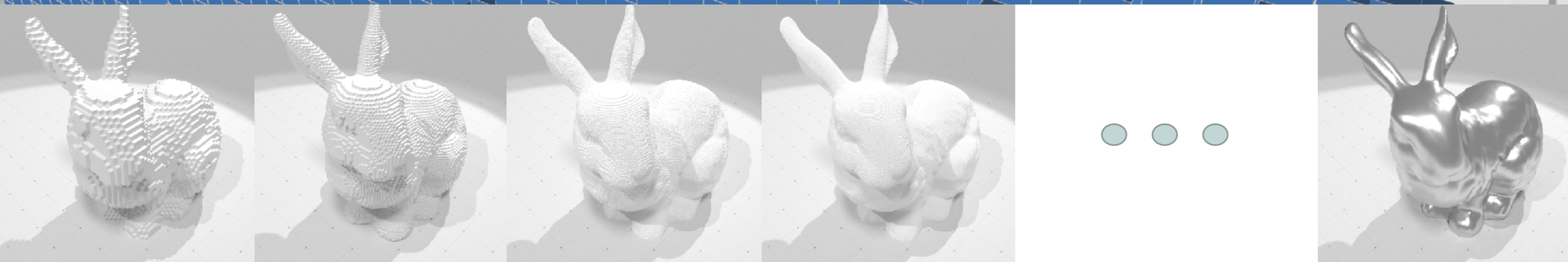


Multigrid convergence (global version)

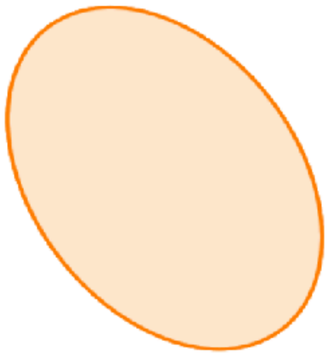
For digitization process G, the discrete geometric estimator $\hat{E}$ is multigrid convergent to the geometric quantity E for the class of shapes $\mathbb{X}$, iff, for any $X \in \mathbb{X}$, there exists a grid step $h_X > 0$, such that :

$\hat{E}(G_h(X), h)$ is defined for any $0 < h < h_X$,

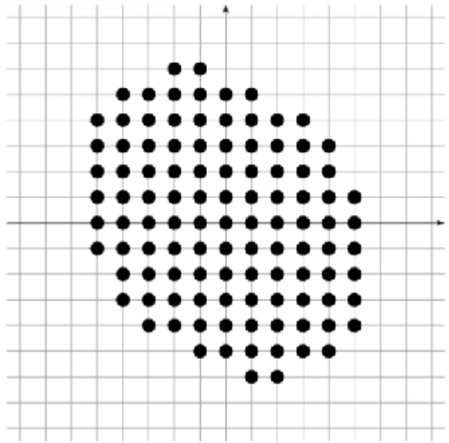
$$|\hat{E}(G_h(X), h) - E(X)| < \tau_X(h)$$

where the speed of convergence $\tau_X(h)$ has null limit when $h \to 0$.

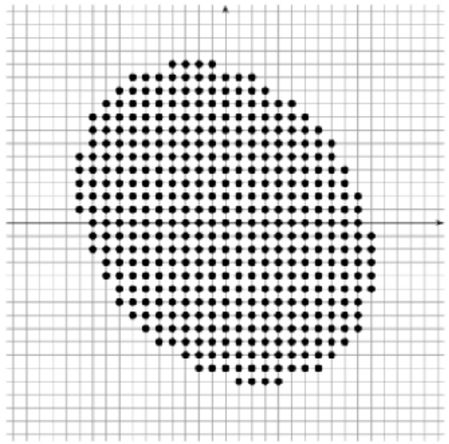
(Typically area, perimeter, integrals)



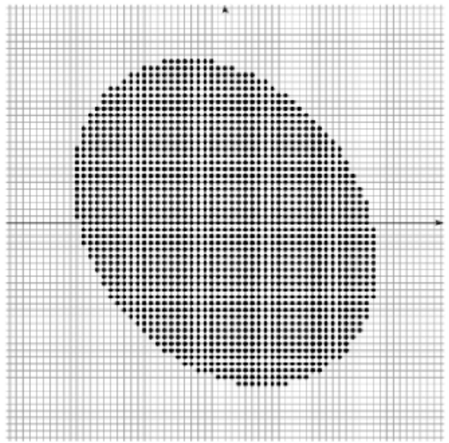
$M \in \mathbb{X}$



$\mathbb{G}_1(M)$



$\mathbb{G}_{0.5}(M)$



$\mathbb{G}_{0.25}(M)$

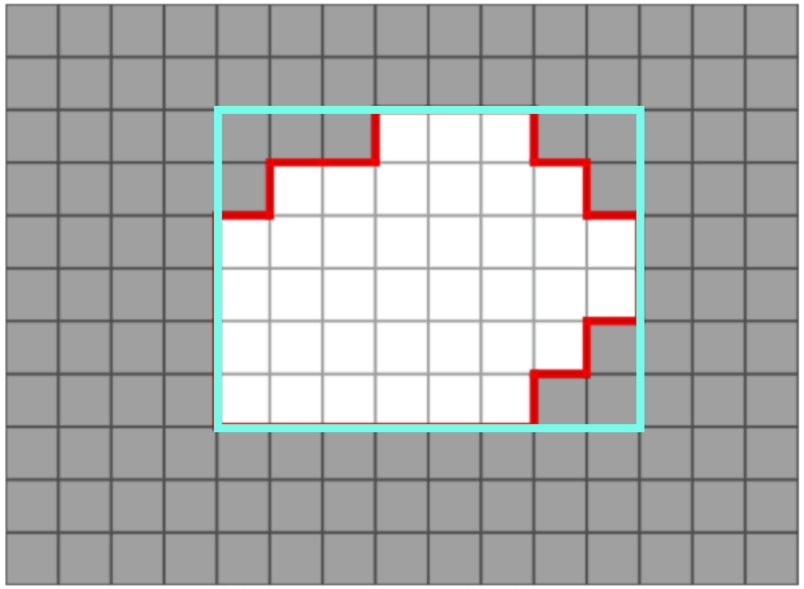
$\widehat{\text{Area}}\,(G_h(X), h) := h^2 \text{Card}(G_h(X))$

tends toward $\text{Area}(X)$ as $h \to 0$

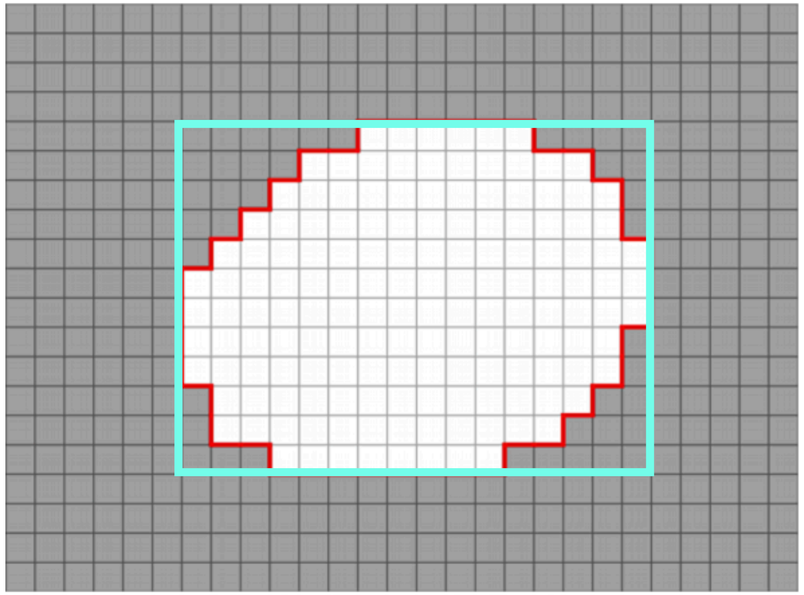
Convergence speed is $O(h)$ [Gauss]

and even $O(h^{\frac{22}{15}})$ for smooth enough ∂X [Huxley 98]

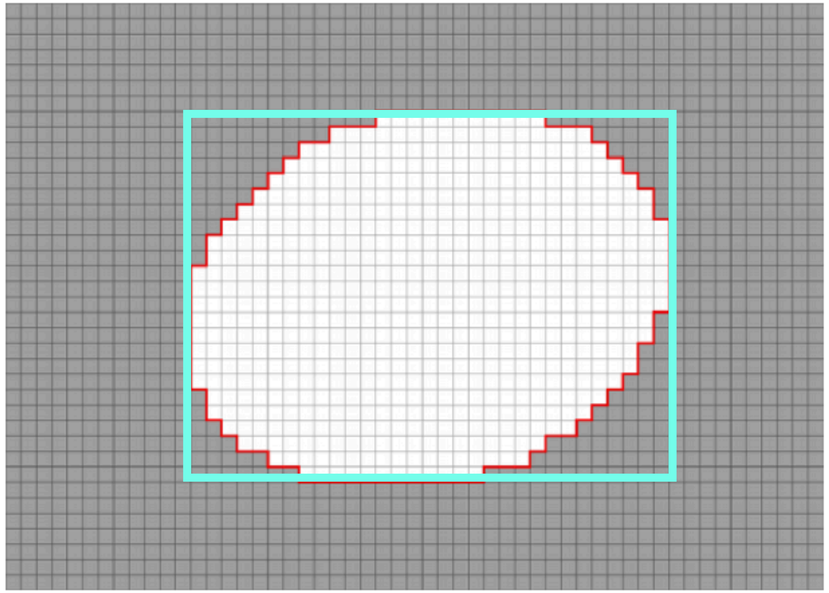
Area by counting is convergent, what about perimeter ?



Digitization of a parametric ellipse h=1

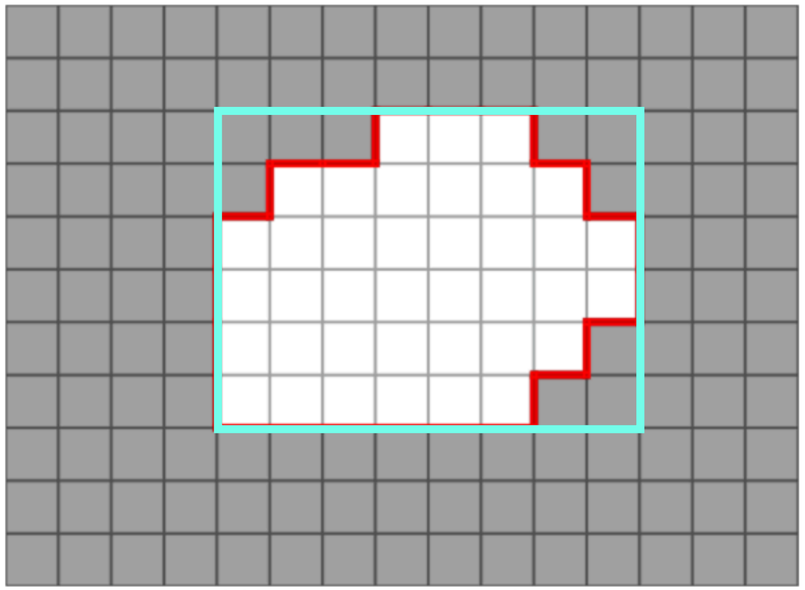


Digitization of a parametric ellipse h=0.5

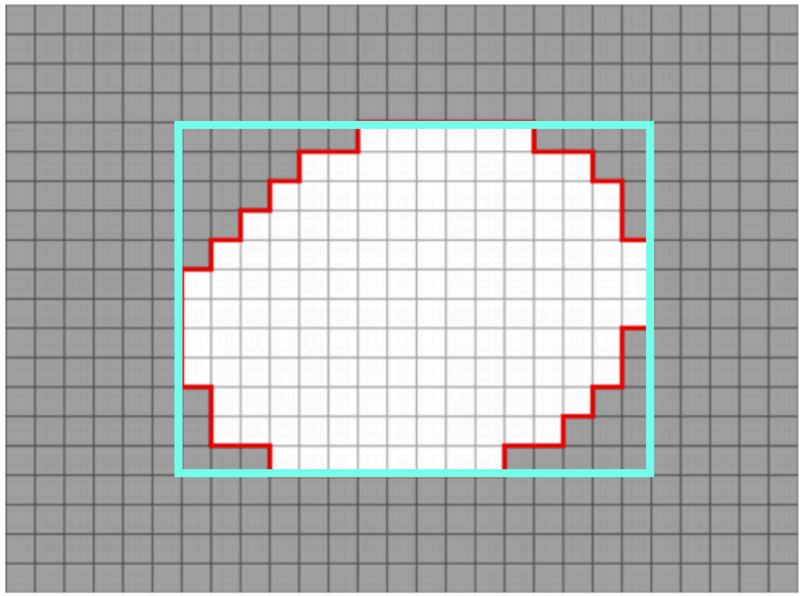


Digitization of a parametric ellipse h=0.25

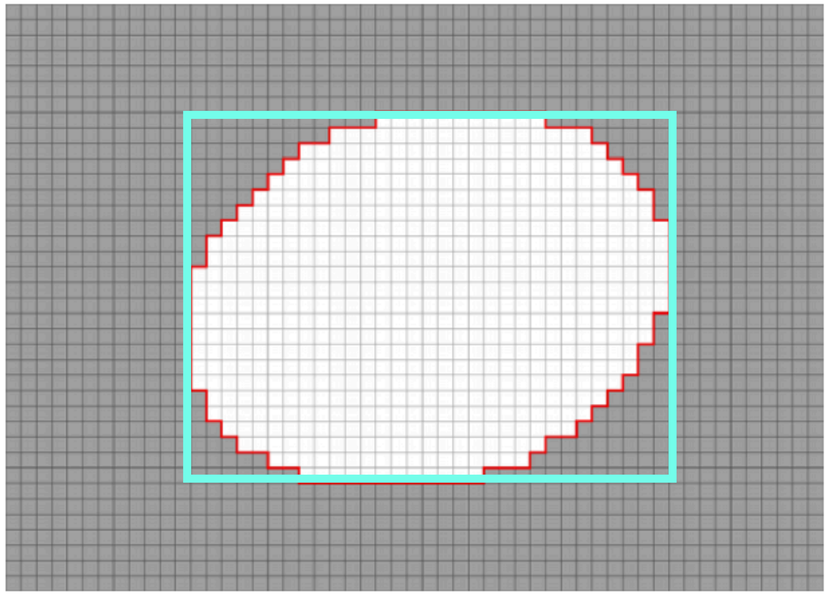
Area by counting is convergent, what about perimeter ?



Digitization of a parametric ellipse h=1



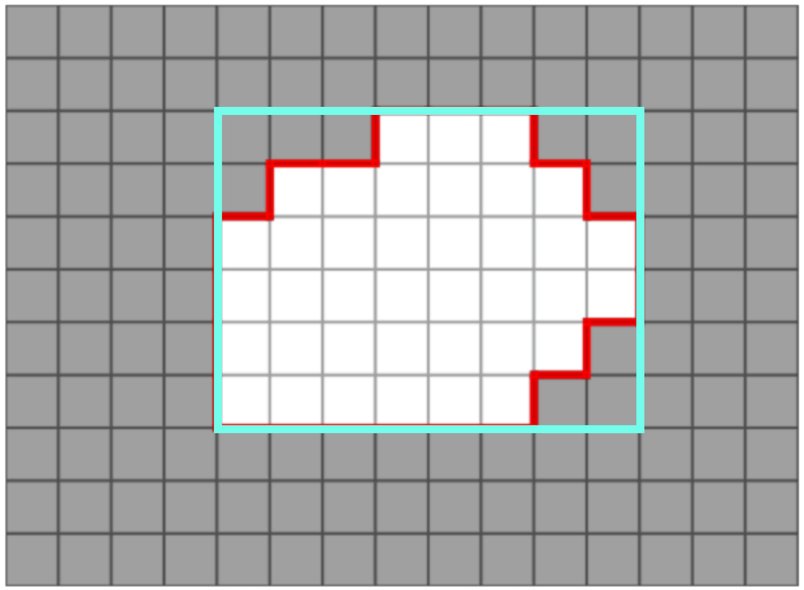
Digitization of a parametric ellipse h=0.5



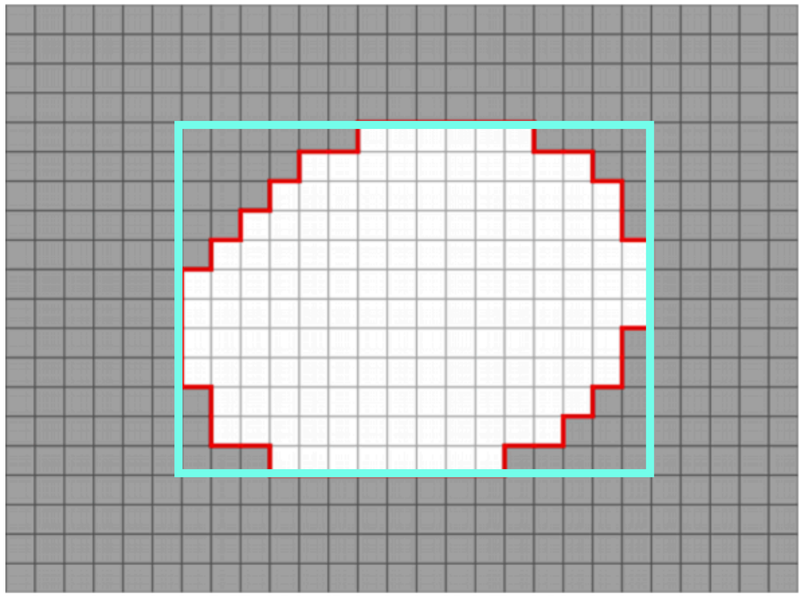
Digitization of a parametric ellipse h=0.25

If counting the number of points on the boundary converges toward the perimeter,
Then $\pi = 4$

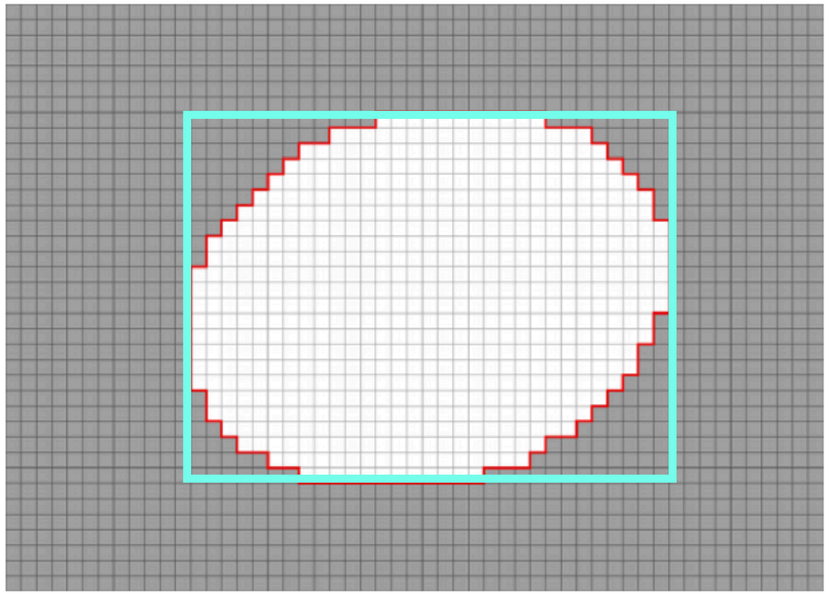
Area by counting is convergent, what about perimeter ?



Digitization of a parametric ellipse h=1



Digitization of a parametric ellipse h=0.5



Digitization of a parametric ellipse h=0.25

If counting the number of points on the boundary converges toward the perimeter,
Then $\pi = 4$

If the shape was convex, computing the perimeter of the convex hull of the digitized shape is convergent, with speed $O(h)$

Multigrid convergence - partial bibliography

2D quantity	shapes	method	max error	reference
volume	C^3-convex	counting	$O(h^{\frac{15}{11}+\epsilon})$	[Huxley 90]
moments	C^3-convex	counting	$O(h^{\frac{15}{11}+\epsilon})$	[Klette,Žunić 00]
perimeter	cvx polygons	polygonalisation	$\approx 4.5h$	[Kovalevsky,Fuchs 92]
perimeter	cvx polygons	"sausaging"	$\approx 5.8h$	[Klette et al. 98]
perimeter	C^3-convex	grid continuum	$\approx 8h$	[Sloboda,Zatko 96]
kth deriv.	C^{k+1}-functions	bin. convolut.	$O(h^{(\frac{2}{3})^k})$	[Malgouyres et al. 08]
kth deriv.	C^{k+1}-functions	Taylor approx.	$O(h^{\frac{1}{k+1}})$	[Provot,Gérard 11]
tangents	C^3-convex	max. segments	$O(h^{\frac{1}{3}})$	[L,de Vielleville,Vialard 07]
3D quantity	shapes	method	max error	reference
normals	$C^{1,1}$-smooth	integral invariant	$O(h^{\frac{2}{3}})$	[L,Coeurjolly,Levallois 17]
normals	$C^{1,1}$-smooth	Vor. cov. meas.	$O(h^{\frac{1}{8}})$	[Cuel,L.,Thibert 14]
curvatures	C^3-smooth	integral invariants	$O(h^{\frac{1}{3}})$	[Coeurjolly,L.,Levallois 13]
curvatures	C^2-smooth	cor. normal. current	$O(h^{\frac{1}{3}})$	[L.Romon,Thibert 22]
dD quantity	shapes	method	max error	reference
volume	convex	counting	$O(h)$	[Gauss,Dirichlet]
volume	C^3-convex	counting	$O(h^k)$, $k < 2$	[Müller 99],[Guo 10]
moments	$C^{1,1}$-smooth	counting	$O(h)$	[L.,Coeurjolly,Levallois 17]
area	$C^{1,1}$-smooth	$\int$ normals	$O(h^{\frac{2}{3}})$	[L.,Thibert 16]
mean curv.	C^2-smooth	varifold	$O(h^{\frac{1}{3}})$ in 3d	[Buet et al. 18]

Multigrid convergence - partial bibliography

Global / Local Quantity

2D quantity	shapes	method	max error	reference
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Multigrid convergence - partial bibliography

Global / Local Quantity

Many class of shapes

2D quantity	shapes	method	max error	reference
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	C^3-convex	counting	$O(h^k)$, $k < 2$	[Müller 99],[Guo 10]
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mean curv.	C^2-smooth	varifold	$O(h^{\frac{1}{3}})$ in 3d	[Buet et al. 18]

Multigrid convergence - partial bibliography

Global / Local Quantity

Different speeds of convergence

Many class of shapes

2D quantity	shapes	method	max error	reference
volume	C^3-convex	counting	$O(h^{\frac{15}{11}+\epsilon})$	
moments	C^3-convex	counting	$O(h^{\frac{15}{11}+\epsilon})$	
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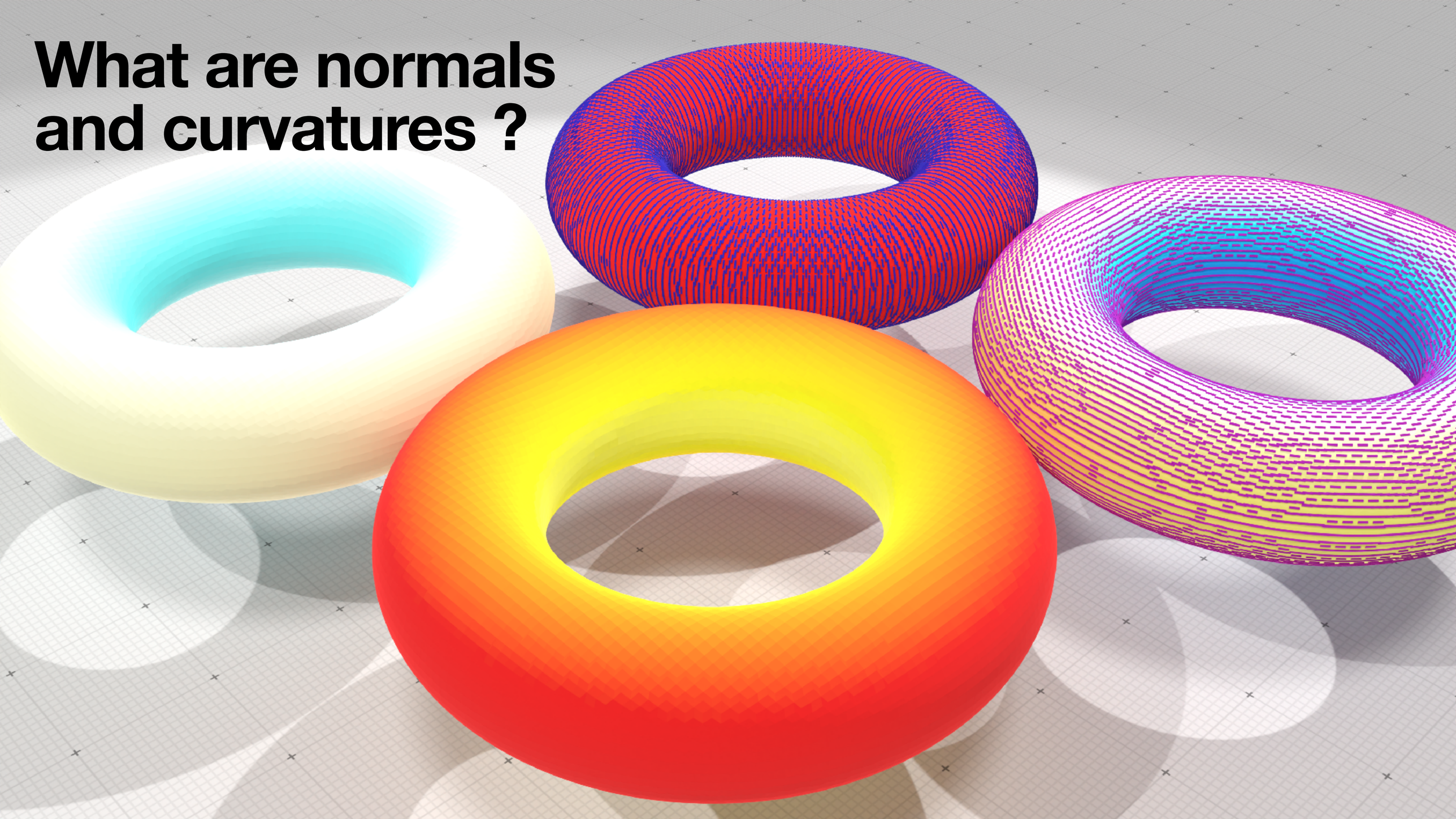
	C^3-convex	counting	$O(h^k)$, $k < 2$	[Müller 99],[Guo 10]
moments	$C^{1,1}$-smooth	counting	$O(h)$	[L.,Coeurjolly,Levallois 17]
area	$C^{1,1}$-smooth	$\int$ normals	$O(h^{\frac{2}{3}})$	[L.,Thibert 16]
mean curv.	C^2-smooth	varifold	$O(h^{\frac{1}{3}})$ in 3d	[Buet et al. 18]

What are normals and curvatures ?



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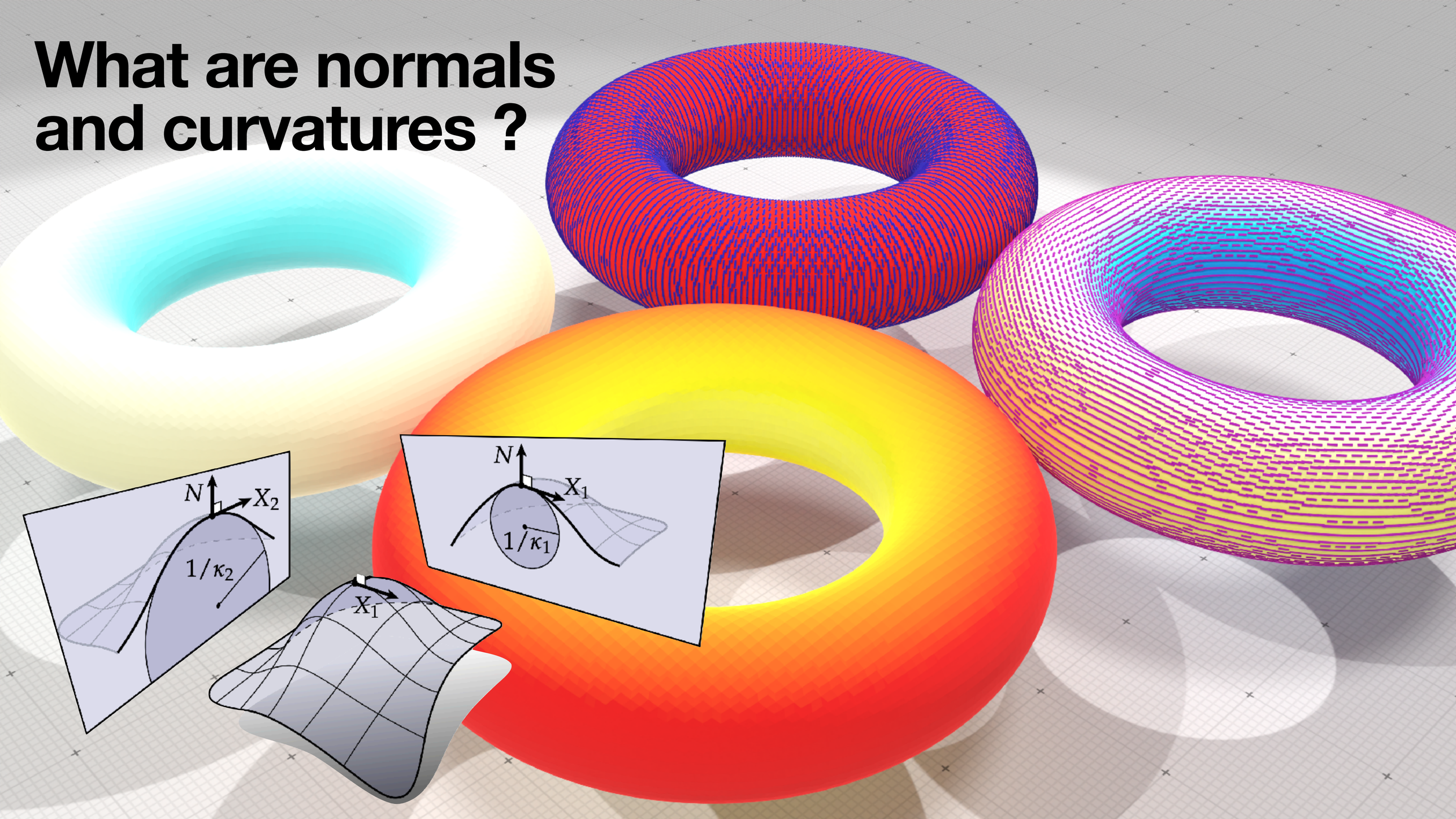

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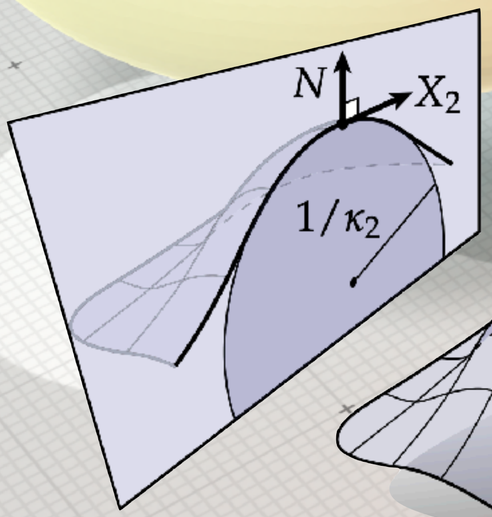
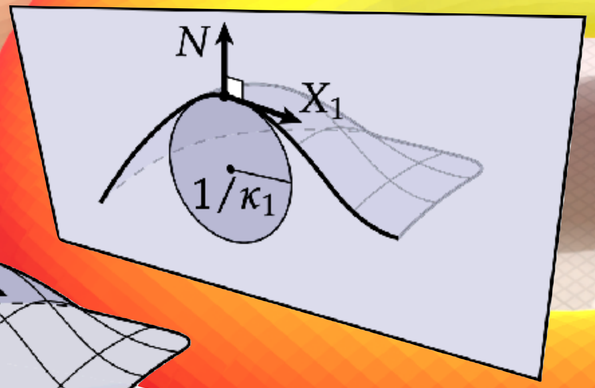
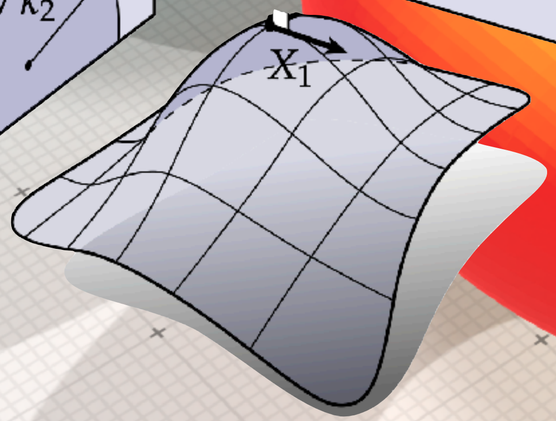


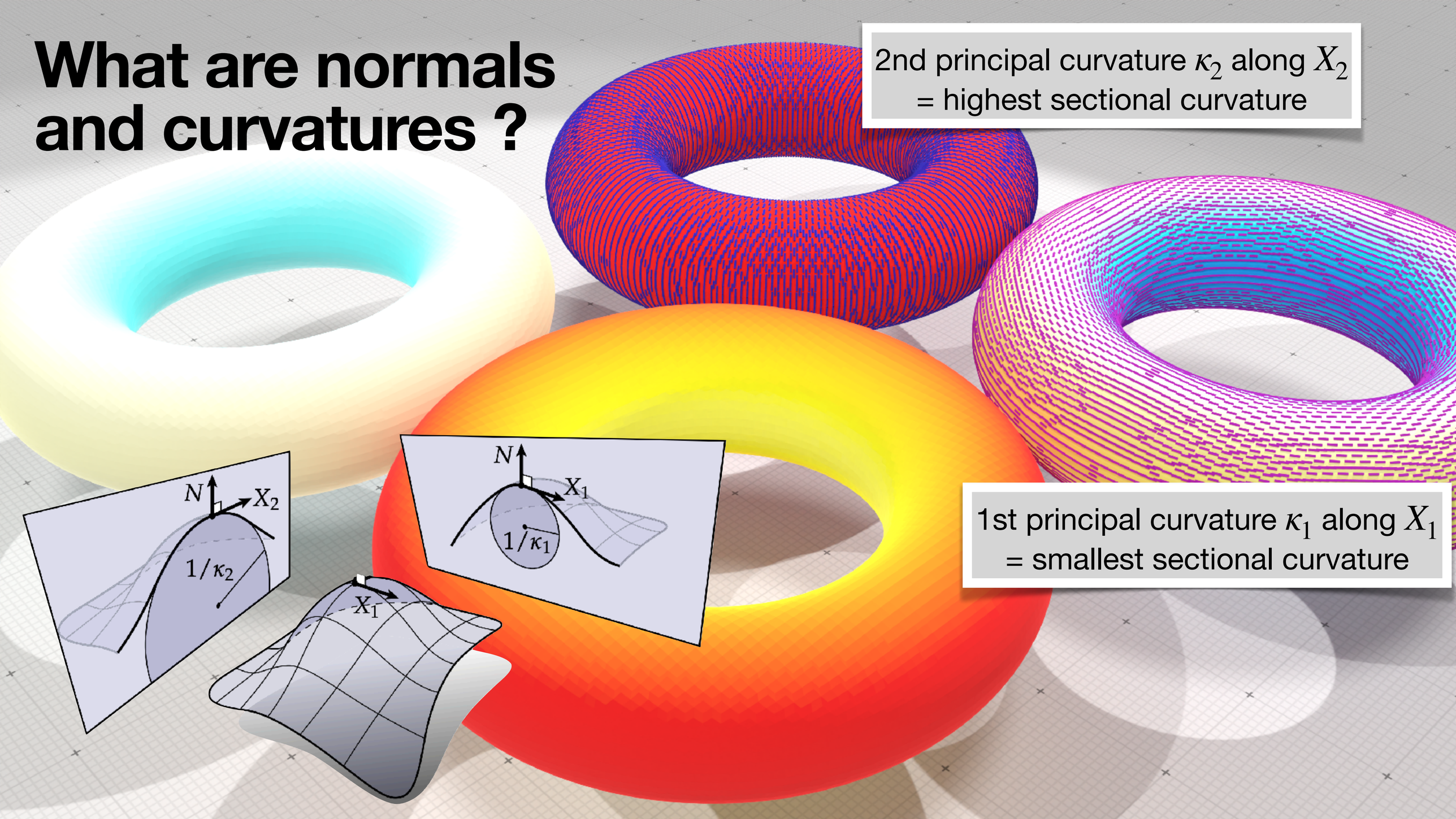




1st principal curvature κ_1 along X_1
= smallest sectional curvature

What are normals and curvatures ?

2nd principal curvature κ_2 along X_2
= highest sectional curvature



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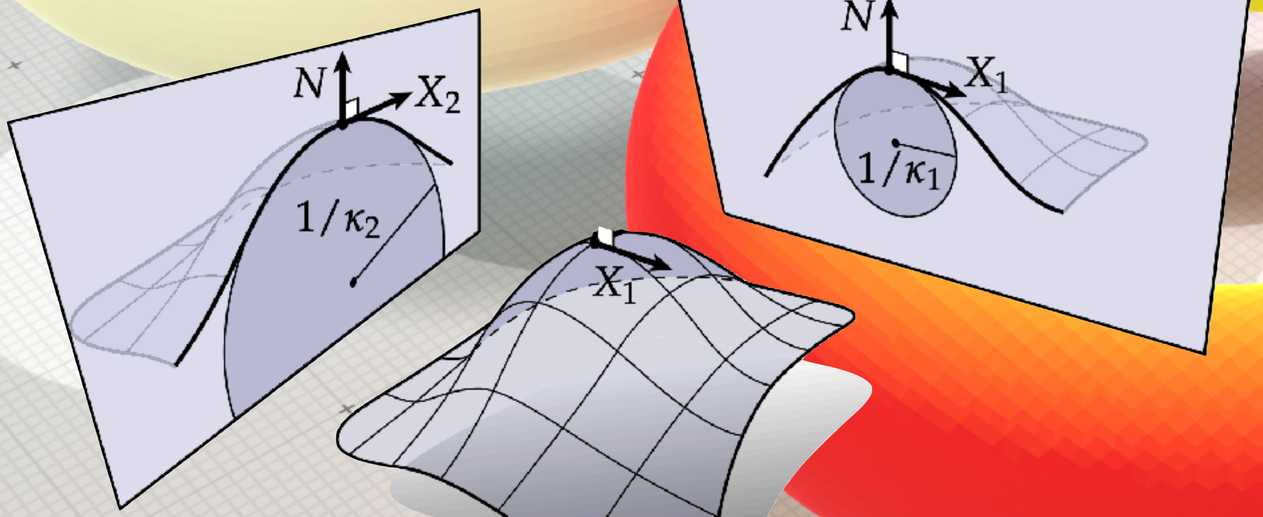

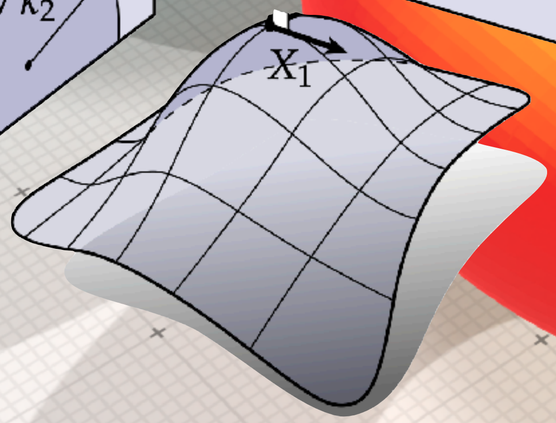

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Mean curvature

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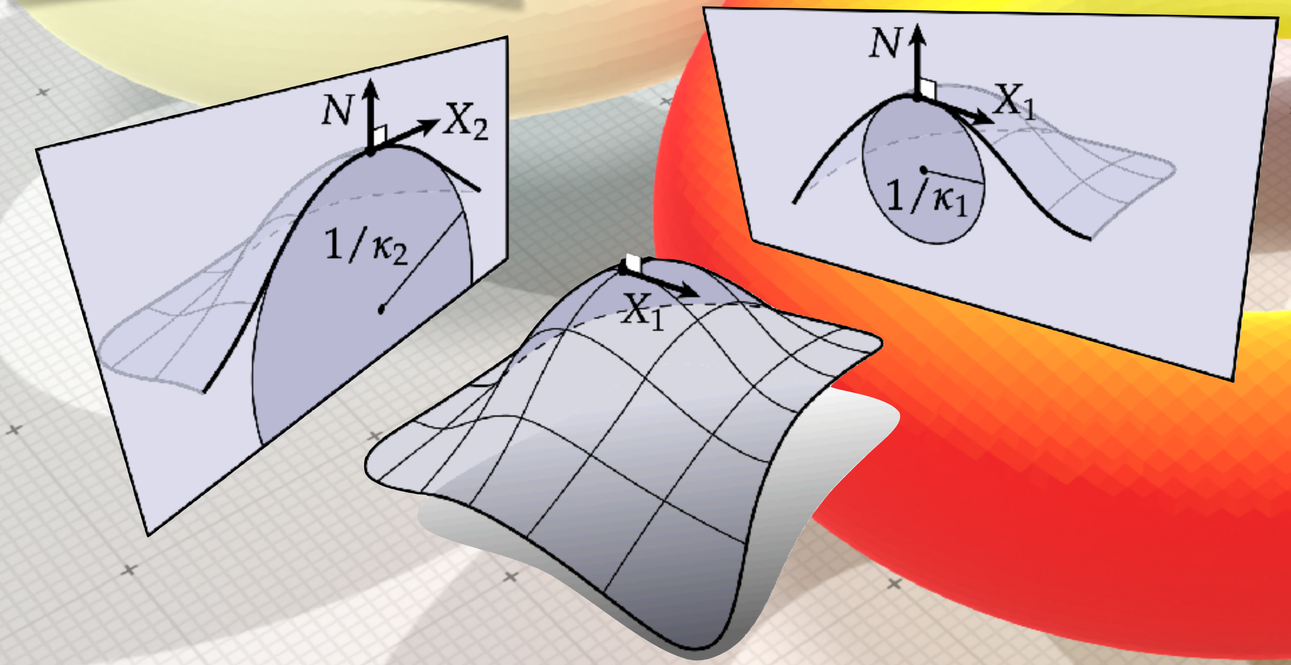
Gaussian curvature
$G := \kappa_1 \kappa_2$



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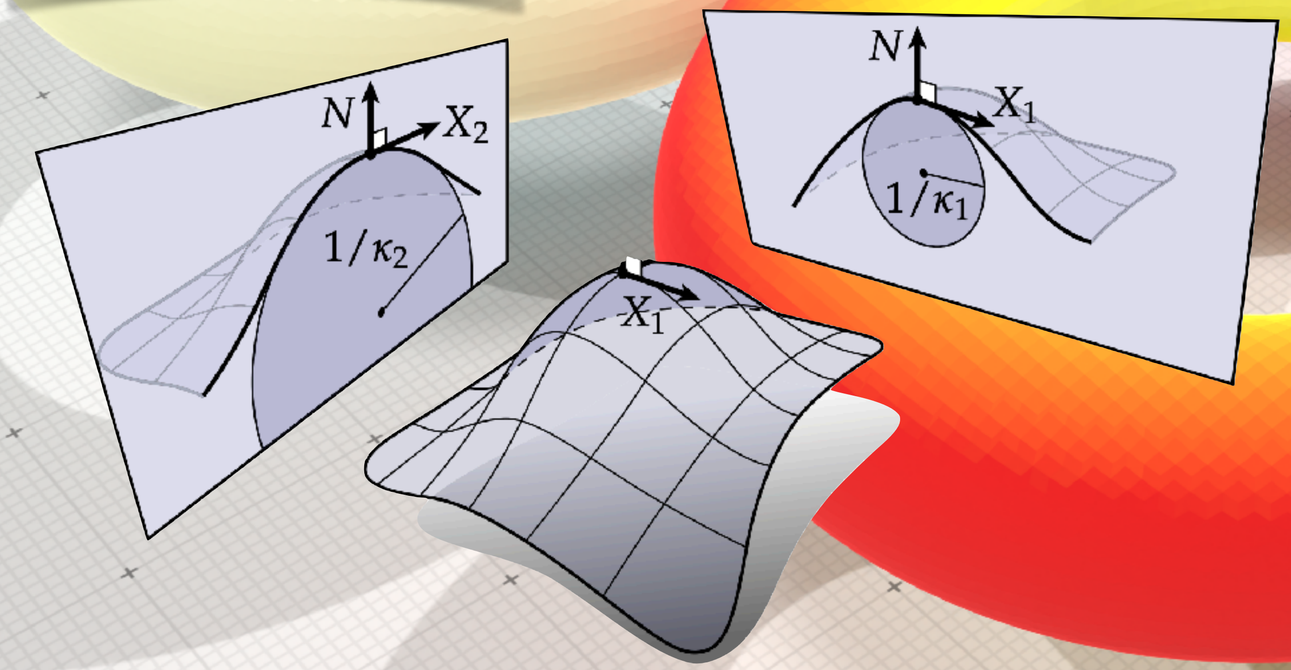

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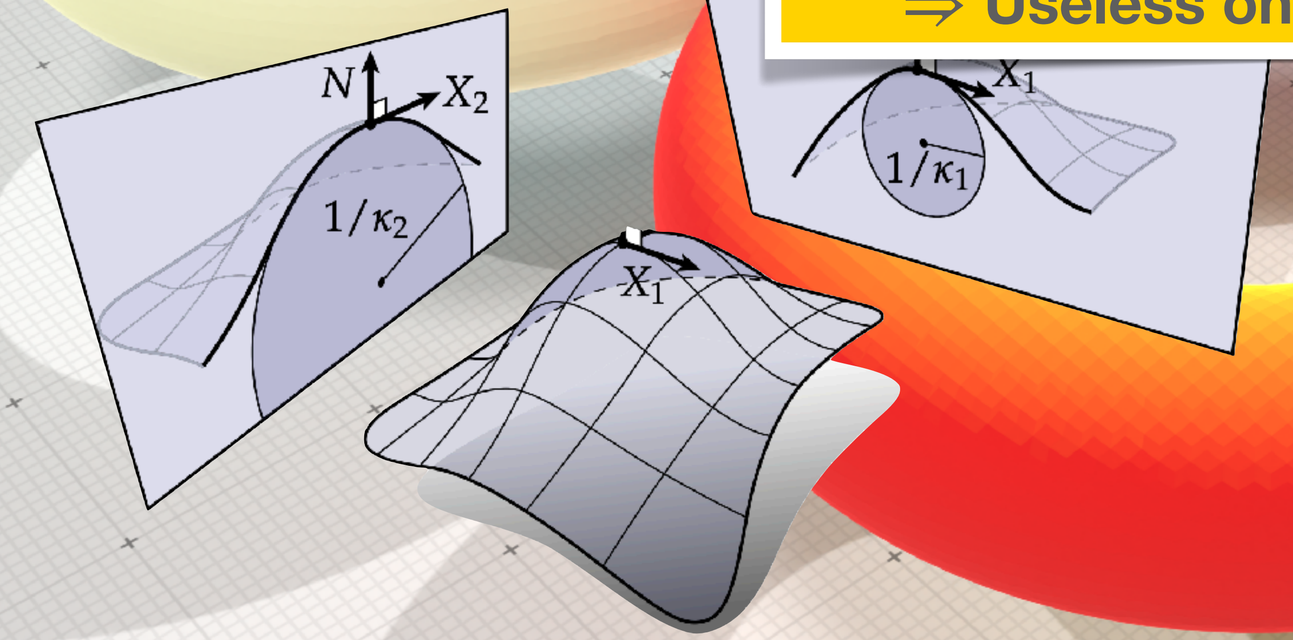
⇒ Useless on digital surfaces



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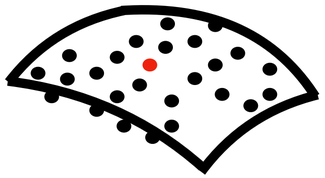
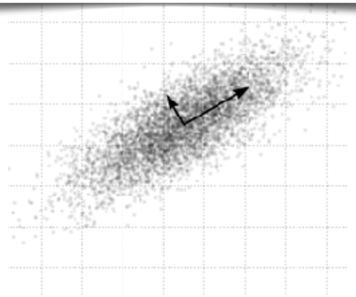
Methods for geometric estimations

Polynomial fitting
Find local best data approximation
In a family of low-dim. polynomials.



Mesh/Volume filtering
Use surface/volume convolutions
with smoothing/derivative kernels

Statistical analysis
Principal component analysis
on lattice points



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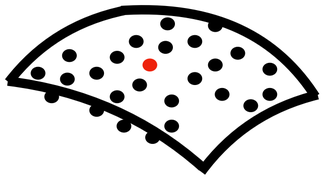

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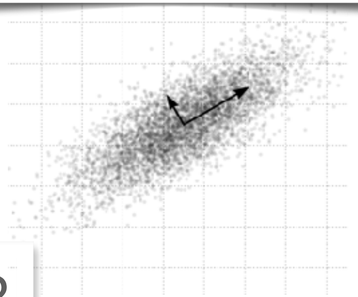

What is the correct degree or window of computation ?

Methods for geometric estimations

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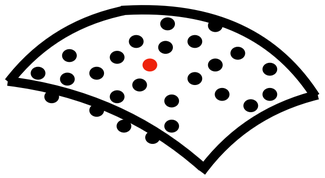

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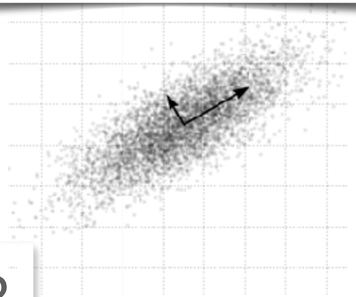
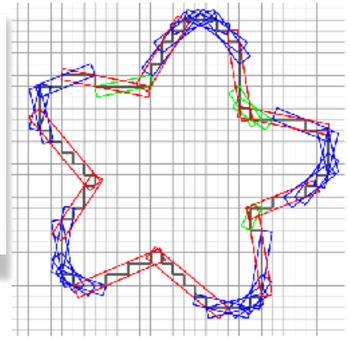


What is the correct degree or window of computation ?

Natural digital window

Digital geometry
Recognition of arithmetic primitives
(digital lines, planes, arc, …)



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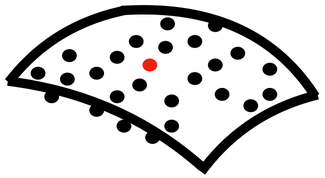

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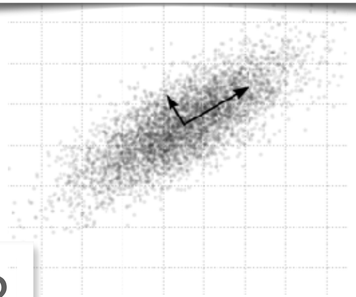

What is the correct degree or window of computation ?

Natural digital window

Optimal radius wrt h

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Recognition of arithmetic primitives
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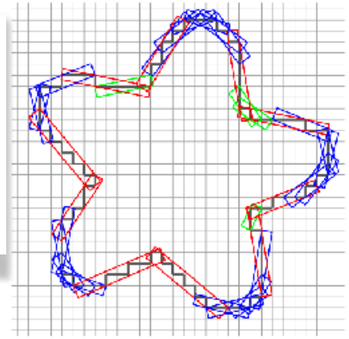
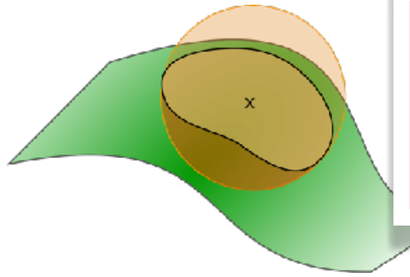
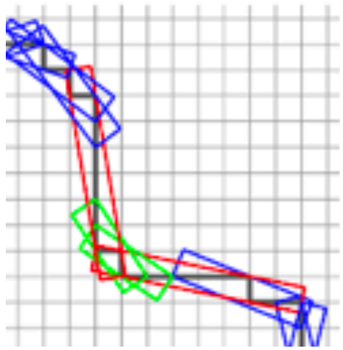


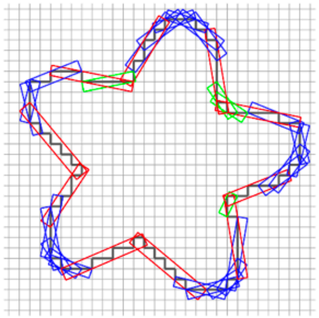


Geometric measure theory
Find volume or surface integrals
stable through digitization

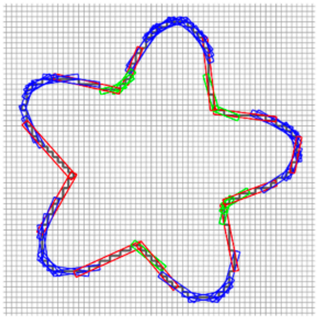
Digital geometry in 2d (parameter-free)



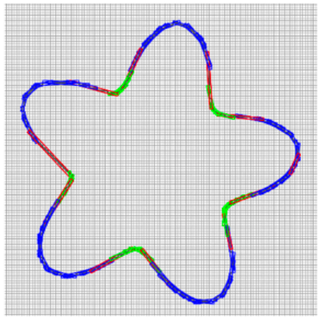
X



$\mathrm{Dig}_1(X)$



$\mathrm{Dig}_{\frac{1}{2}}(X)$



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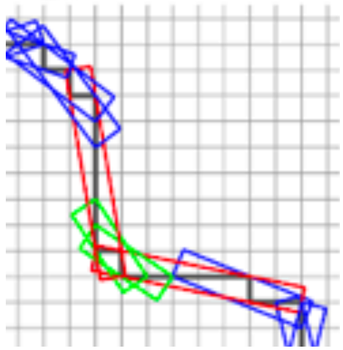
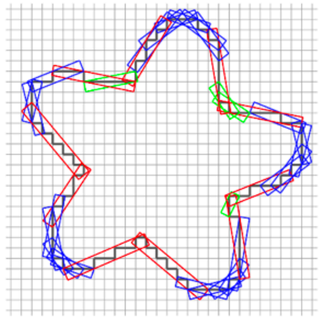
[de Vieillevielle, L, Feschet 2005]



X



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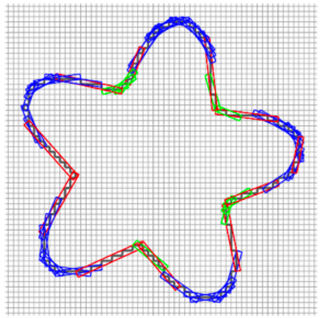

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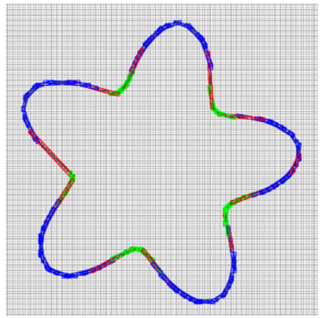

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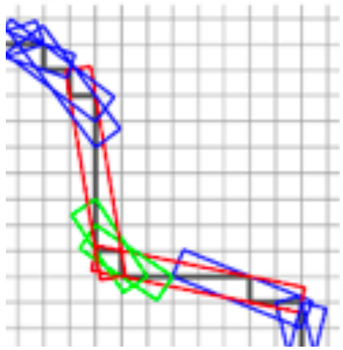
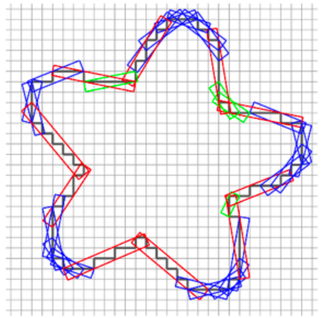
#Vertices of lattice polytopes

[Balog, Barany 91]



X



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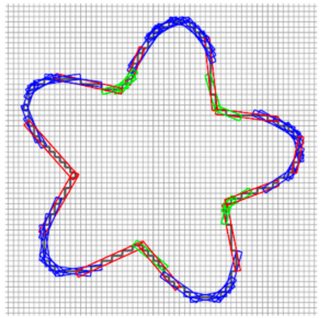

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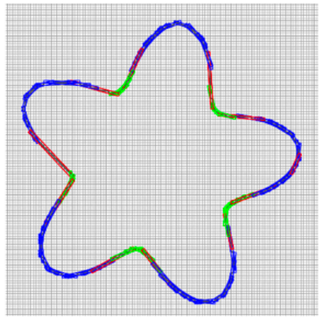

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simple continued fractions

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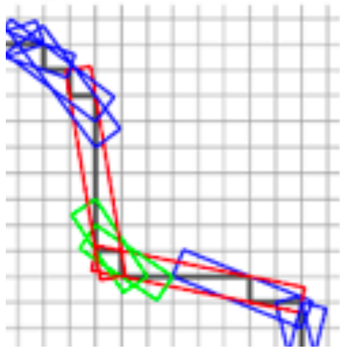
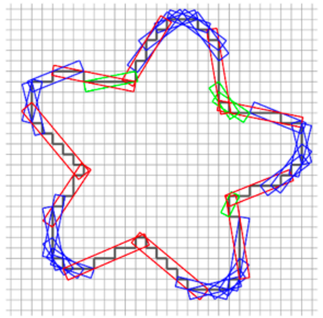


X



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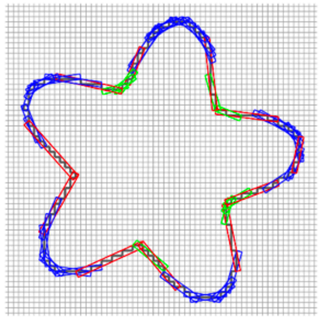

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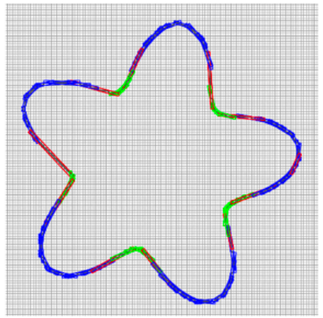

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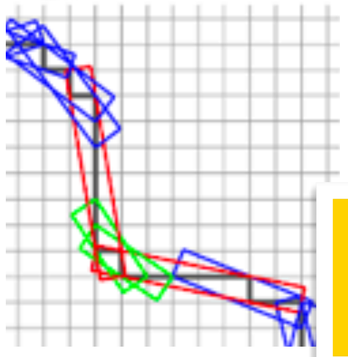
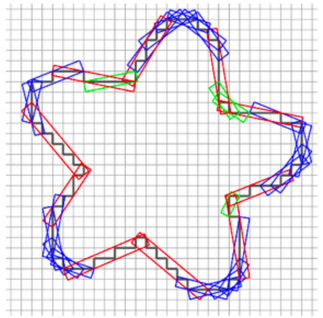


#Vertices of lattice polytopes
[Balog, Barany 91]

MDSS Tangent estimation in $O(h^{\frac{1}{3}})$, linear time
[L, de Vieilleville, Vialard 2007]

X



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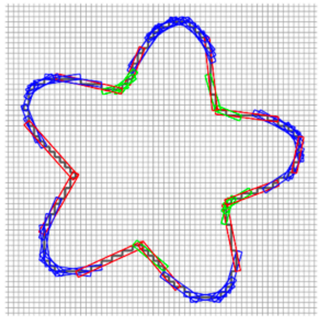

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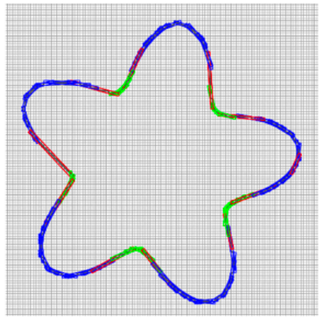

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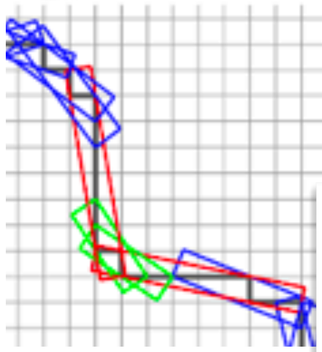

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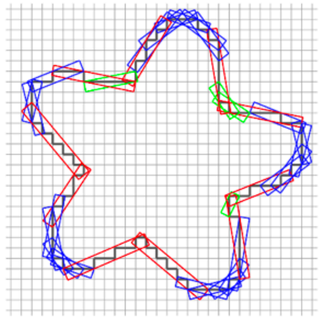
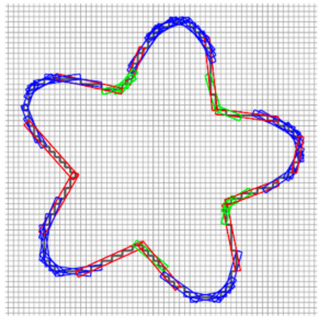
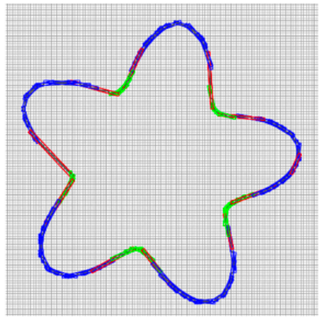


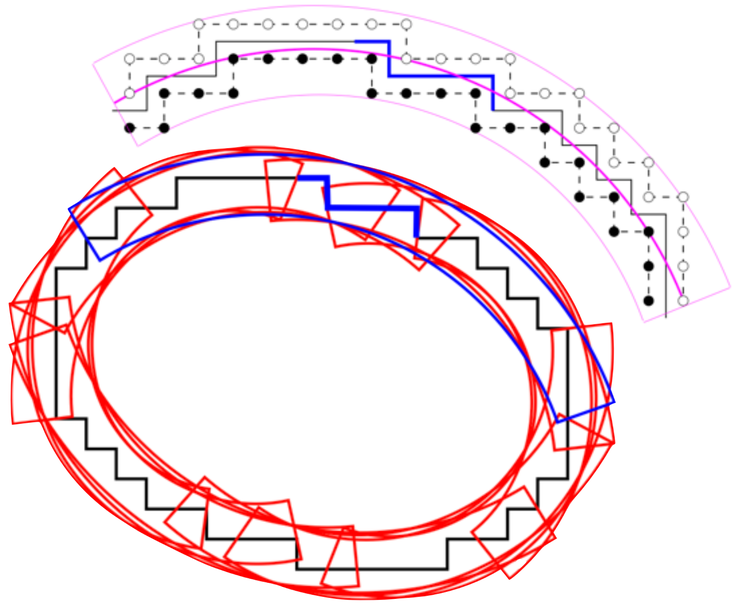
$\mathrm{Dig}_1(X)$



$\mathrm{Dig}_{\frac{1}{2}}(X)$



$\mathrm{Dig}_{\frac{1}{4}}(X)$



[Roussillon, L. 2011]

Digital geometry in 2d (parameter-free)

Avg length of max. digital straight segments follow:

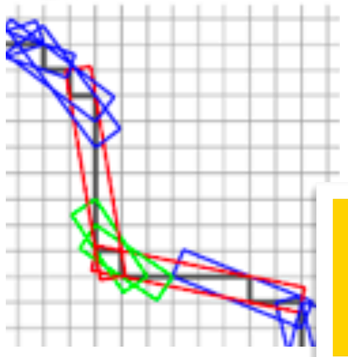
$$\Theta(h^{\frac{2}{3}}) \leq \bar{L}_{MS}(\partial_h X) \leq \Theta(h^{\frac{2}{3}} \log \frac{1}{h})$$

[de Vieillevielle, L, Feschet 2005]

simple continued fractions

Curvature estimation with maximal circular arcs in $O(h^{\frac{1}{3}})$

[Schindele, Massopust, Forster 2017]



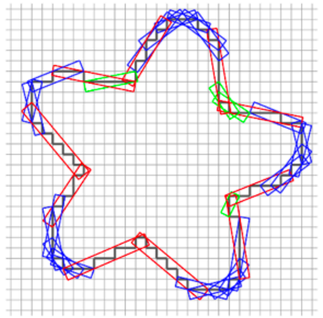
#Vertices of lattice polytopes

[Balog, Barany 91]

MDSS Tangent estimation in $O(h^{\frac{1}{3}})$, linear time

[L, de Vieilleville, Vialard 2007]

X



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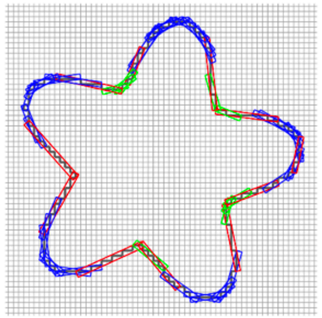

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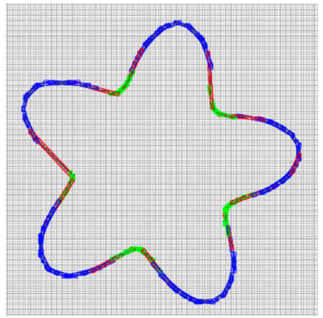

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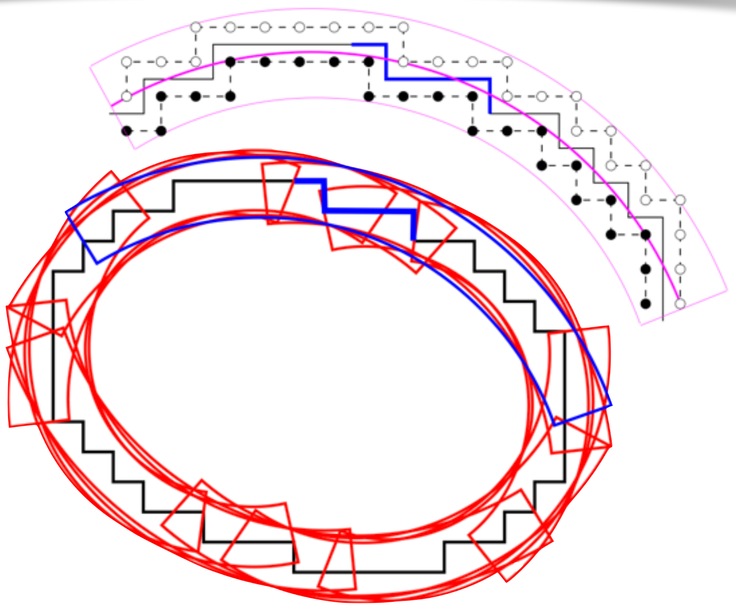
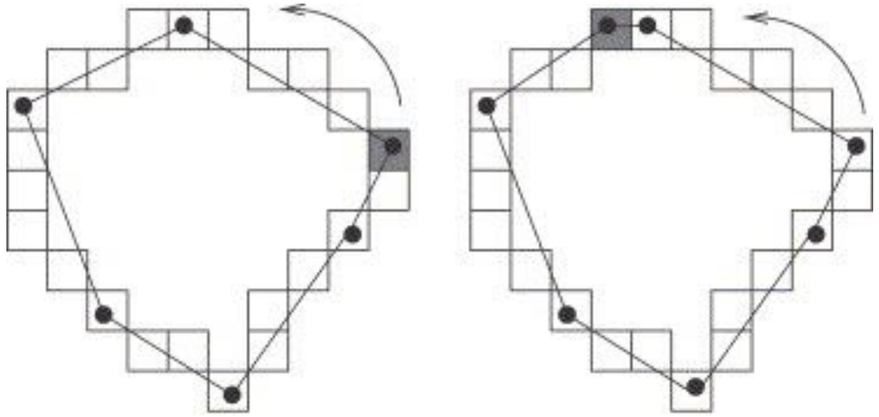


[Roussillon, L. 2011]

Splitting surfaces is difficult

Minimum decomposition of 2d digital contour in maximal digital segments can be done in linear time.

[Feschet, Tougne 2005]



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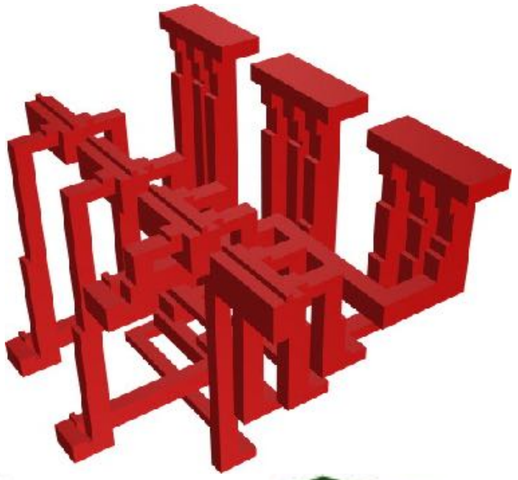

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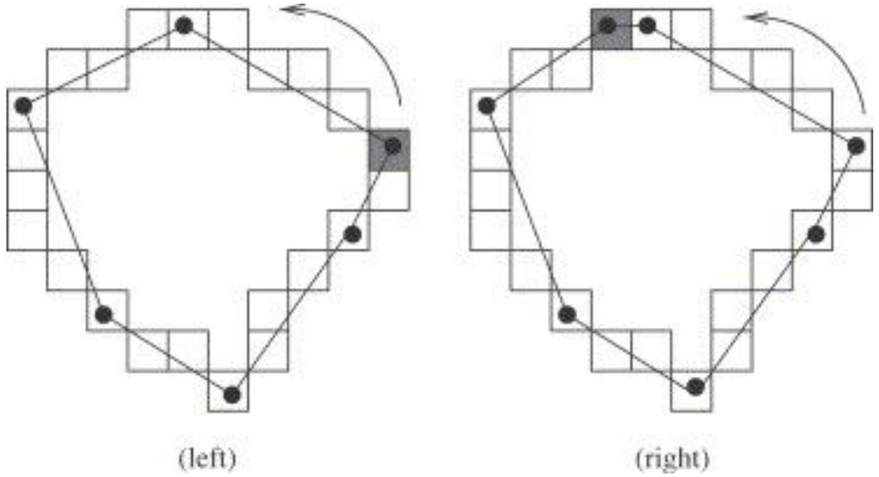
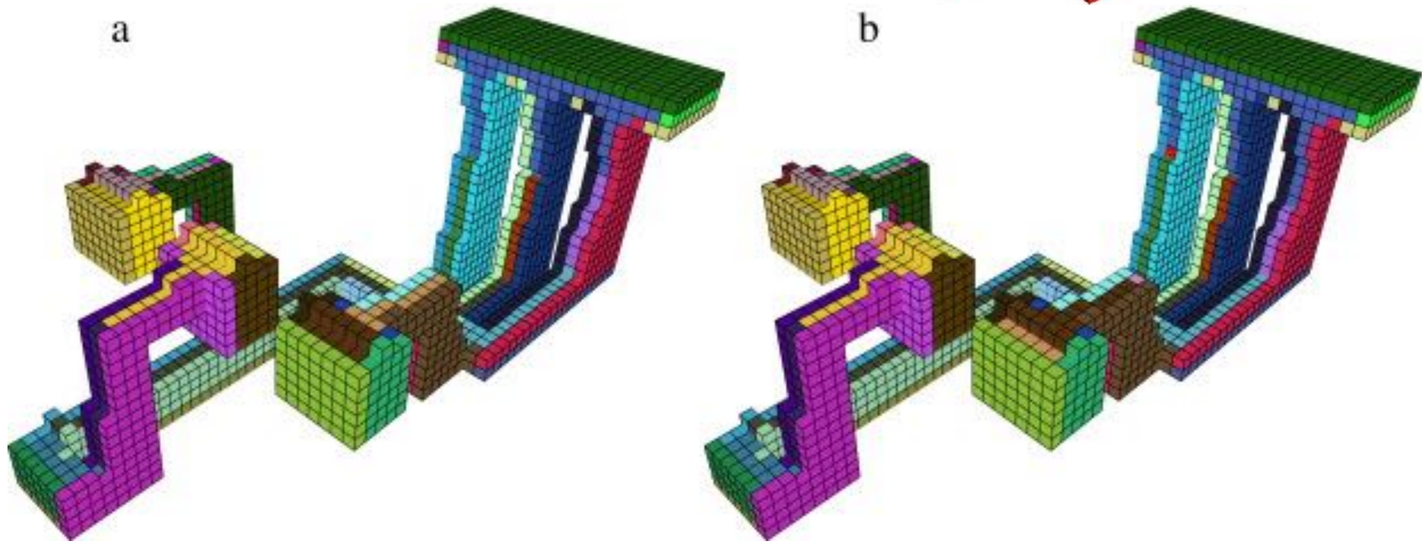
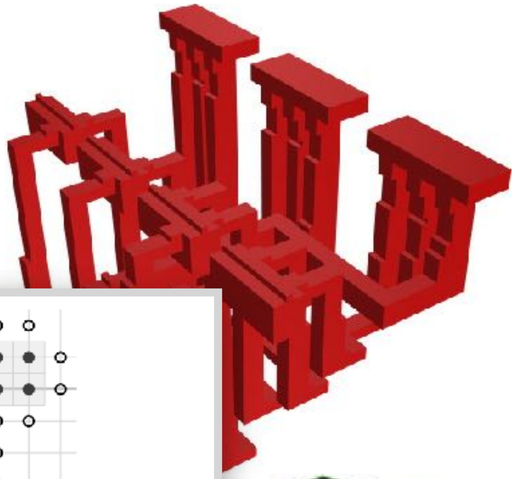
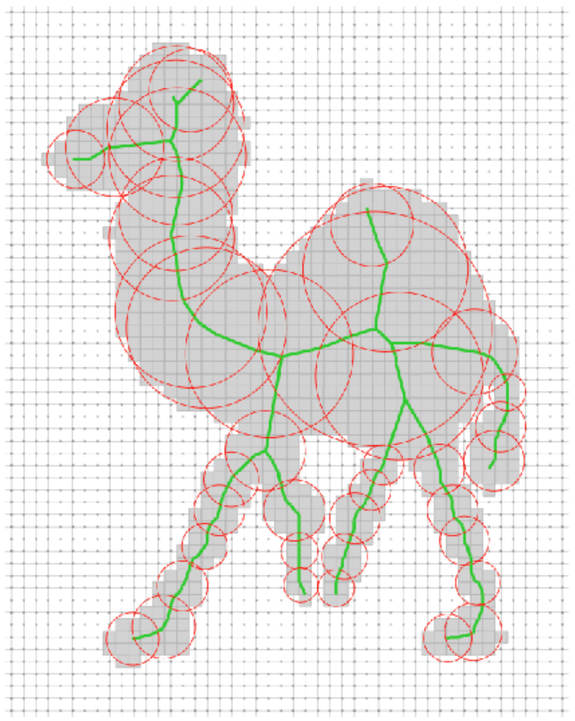


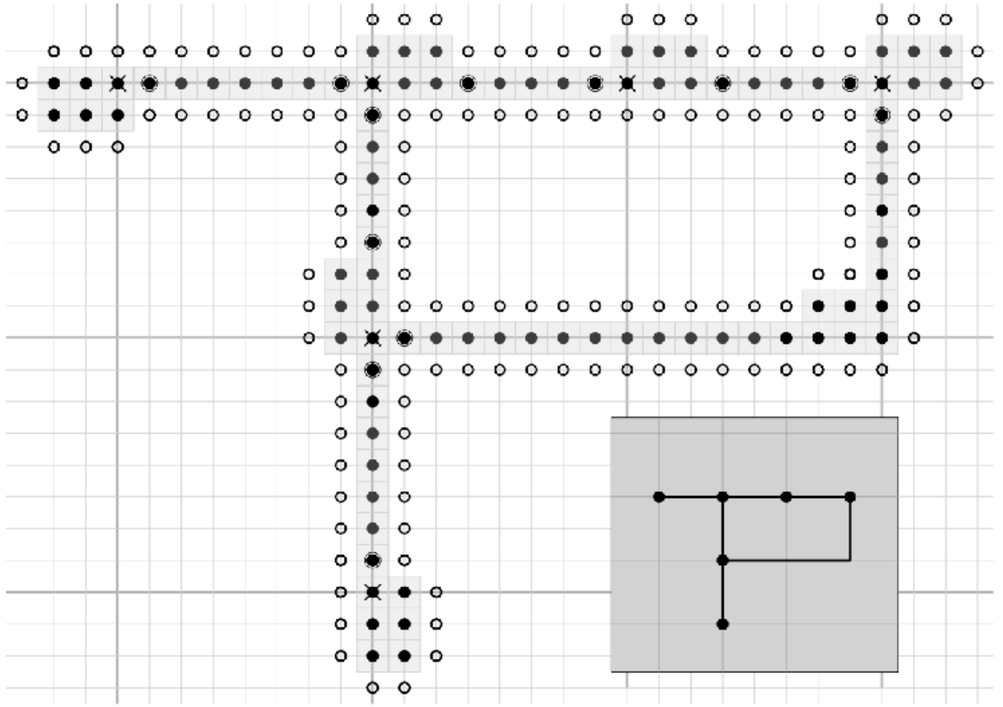


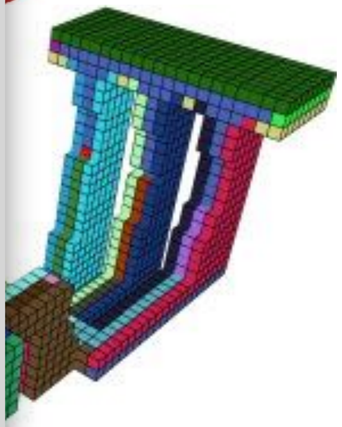
Min-decomposition of a 3d surface in maximal digital planes is NP-hard, by reduction to 3-SAT.
[Sivignon, Coeurjolly 2009]

Splitting surfaces is difficult









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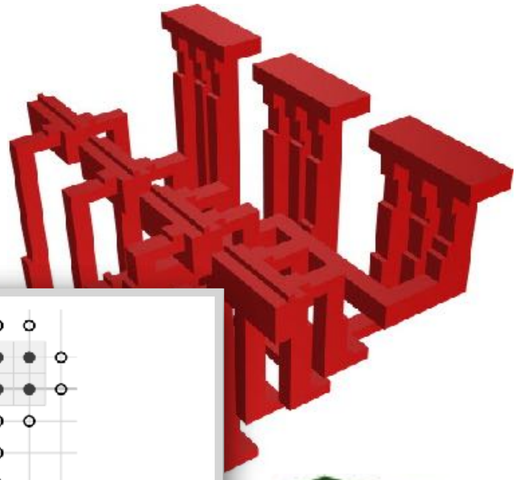
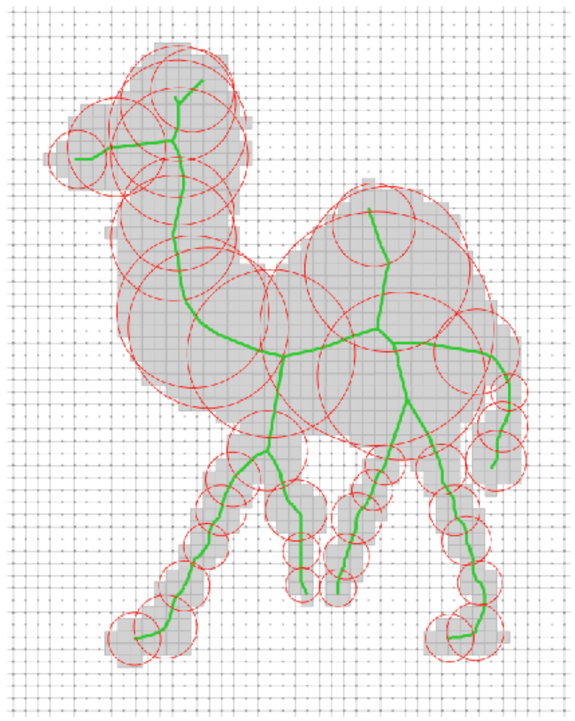
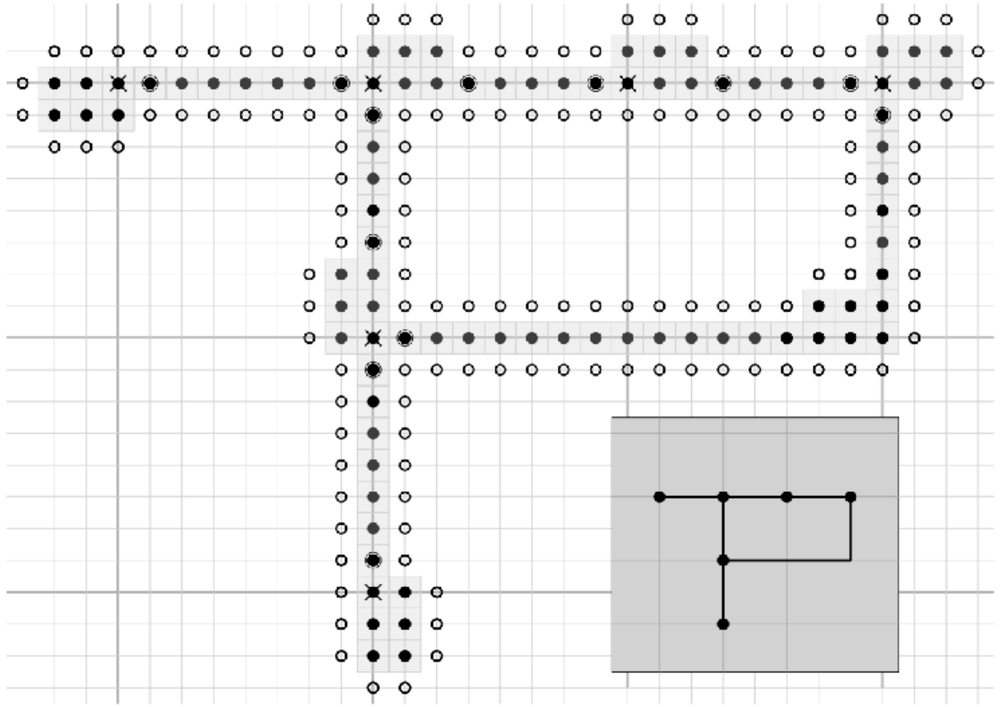
Minimum covering by balls in 2D is **polynomial** if shape is **hole-free**
Otherwise it is **NP-complete** by reduction to 3-planar vertex cover.
[Sivignon 22, Sivignon 25]

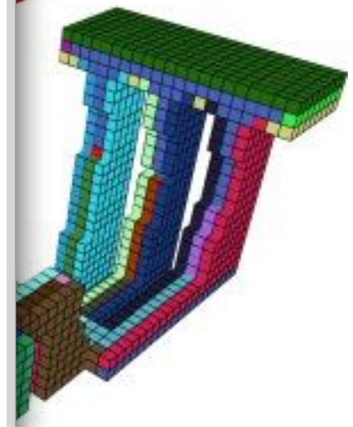
maximal
o 3-SAT.

Splitting surfaces is difficult









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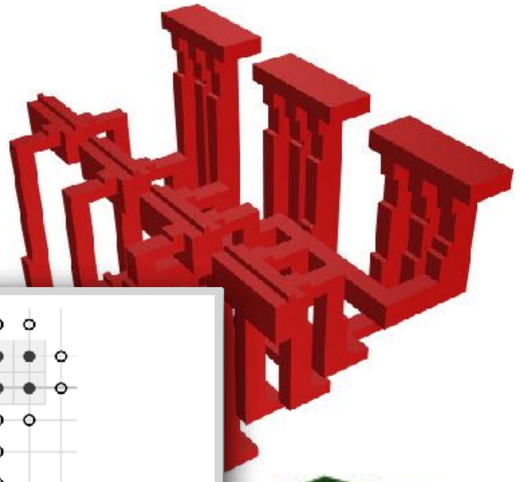

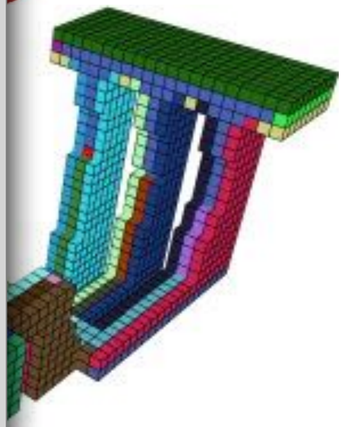

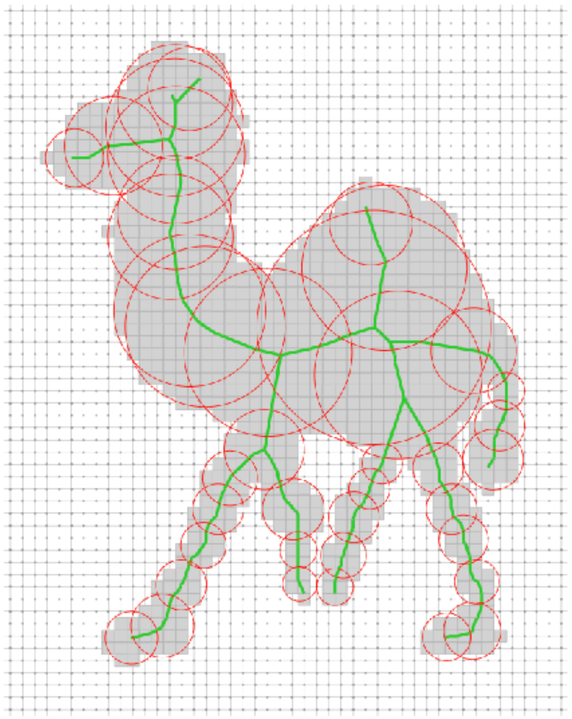

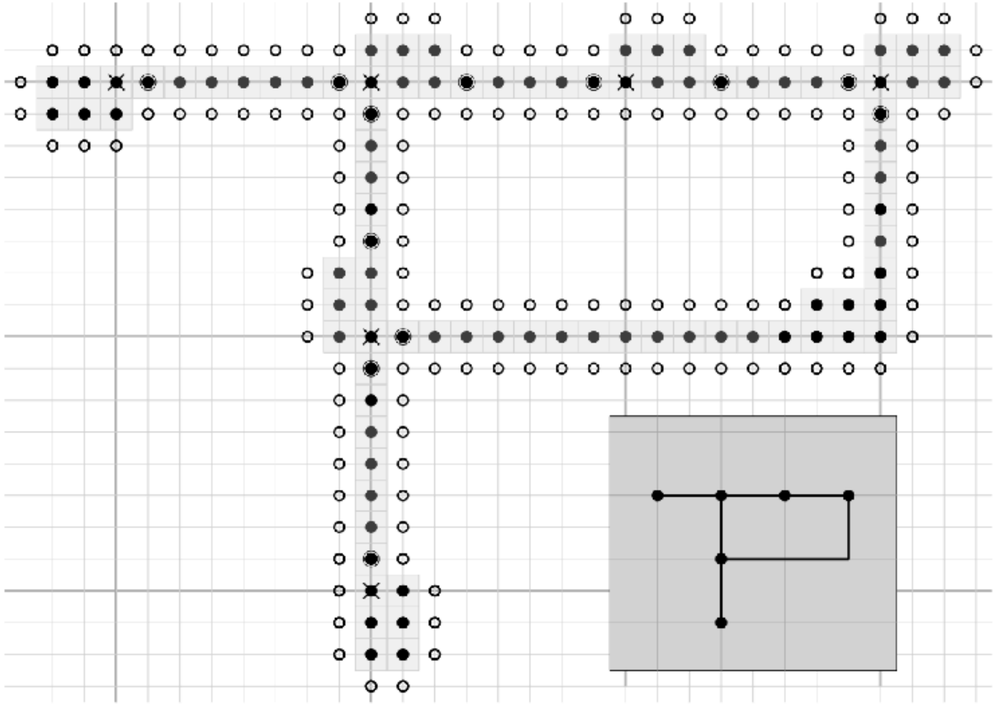

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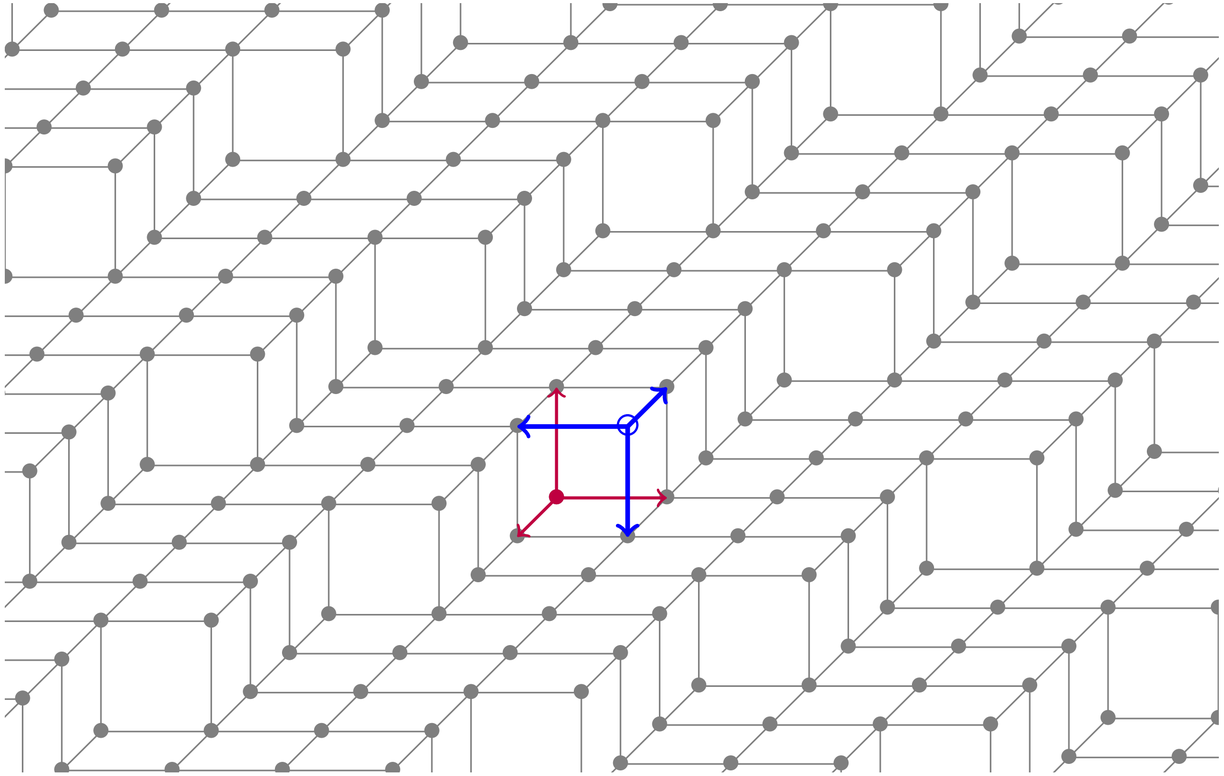
Digital geometry in 3d (parameter-free)

Local estimation of normal vector with successive sparse probings

[L., Provençal, Roussillon 15-24]



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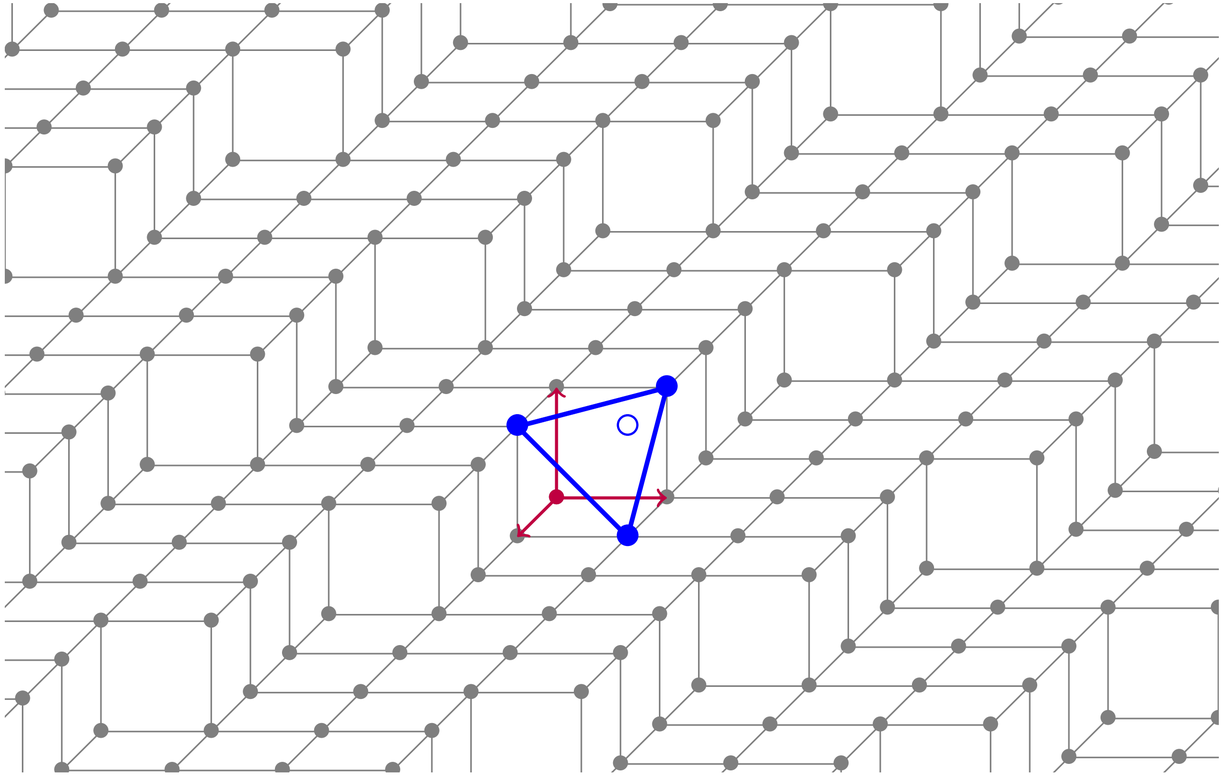

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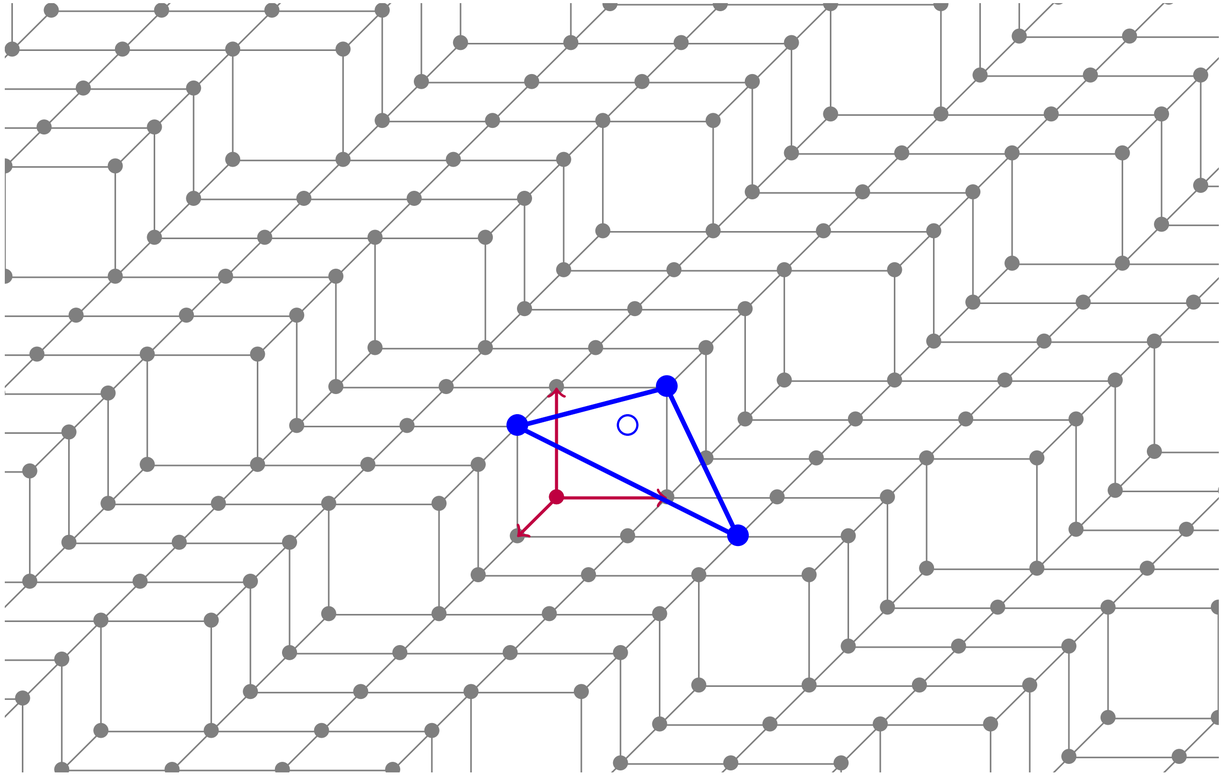

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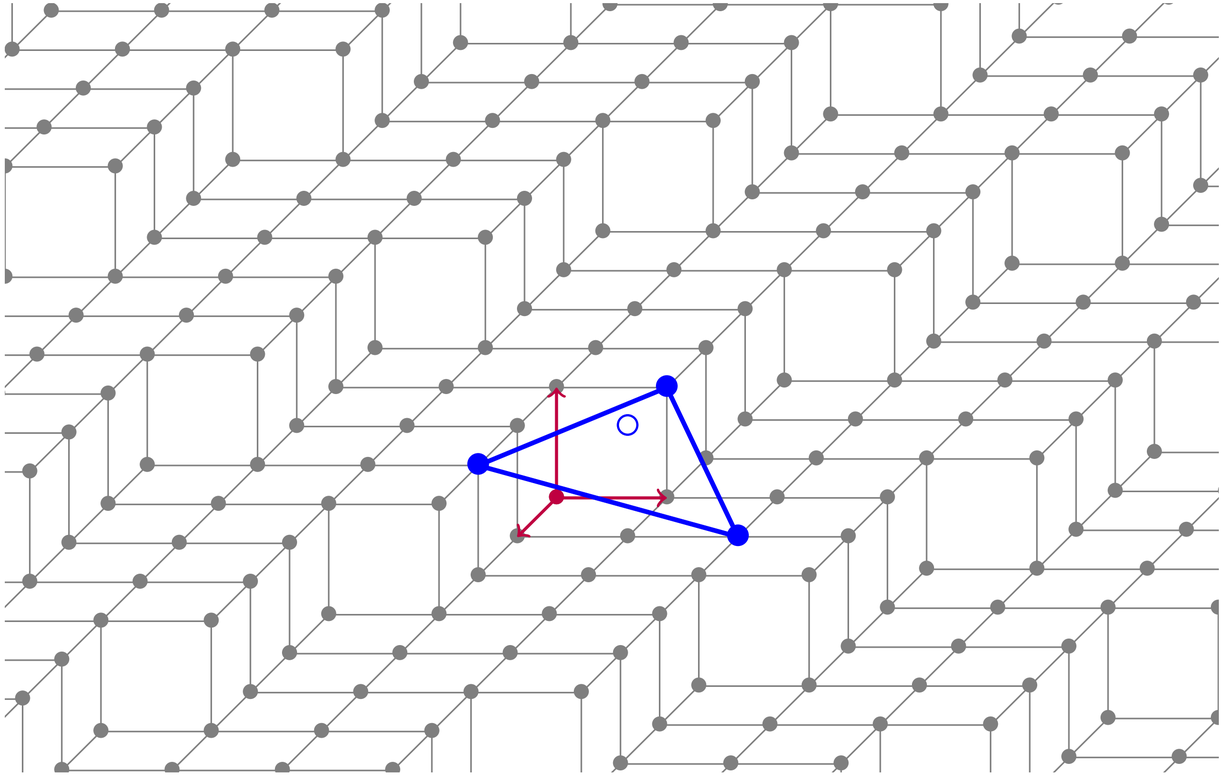

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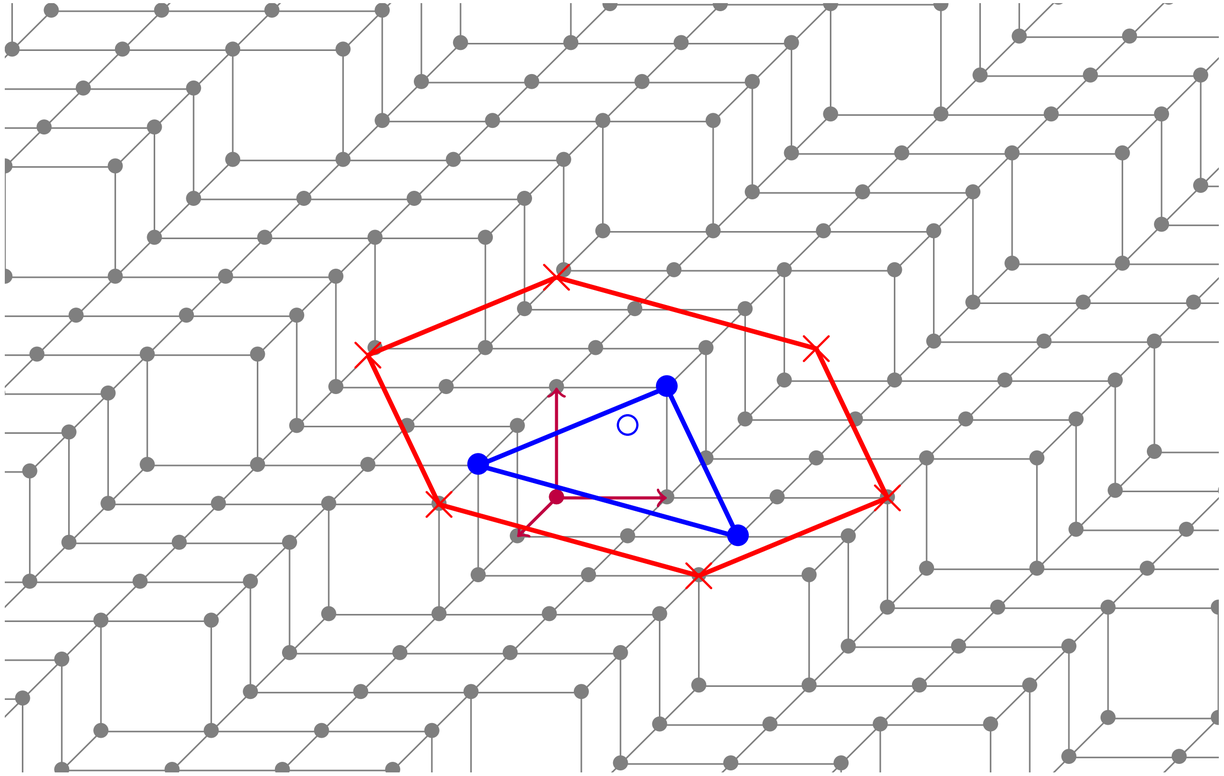

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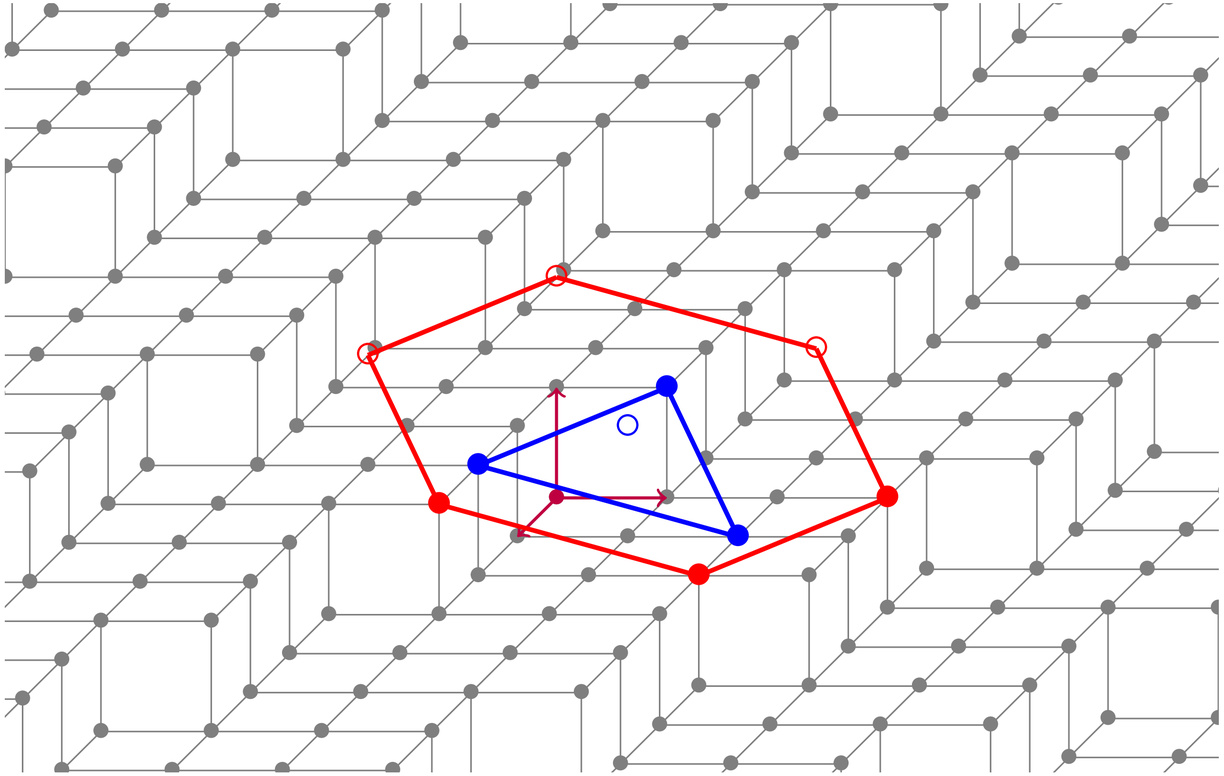

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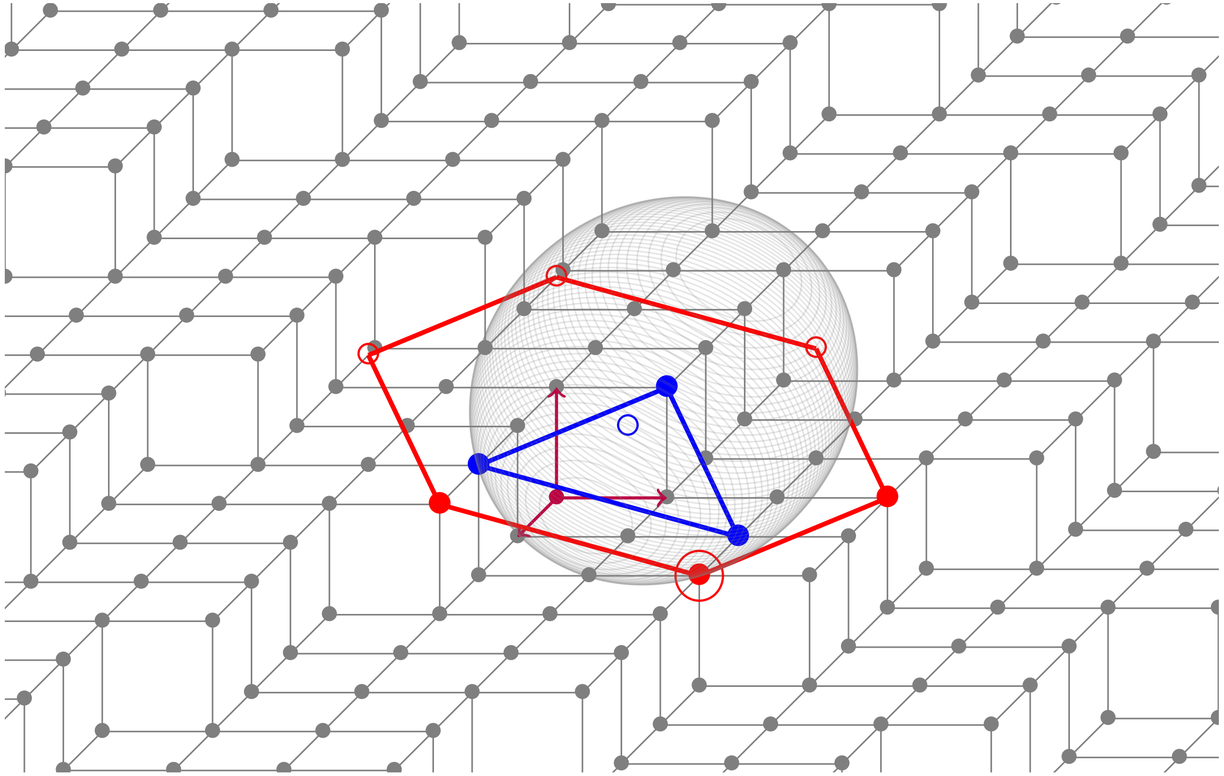

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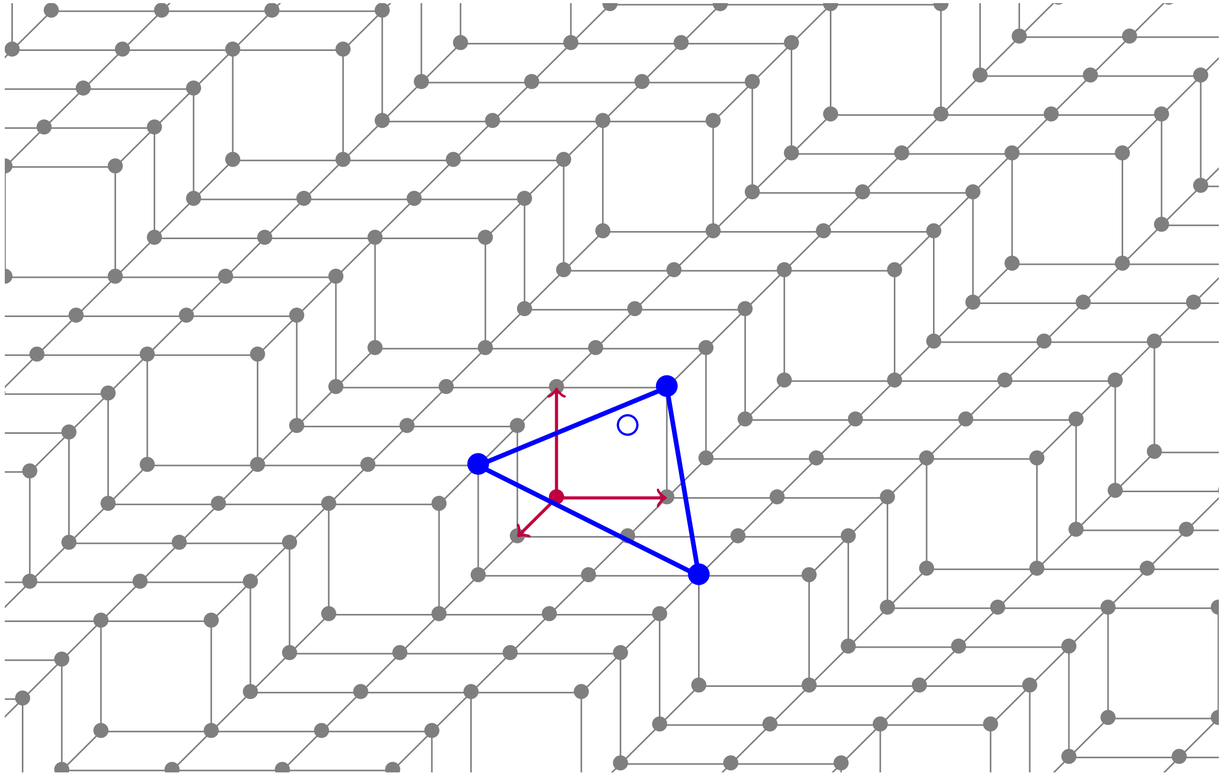

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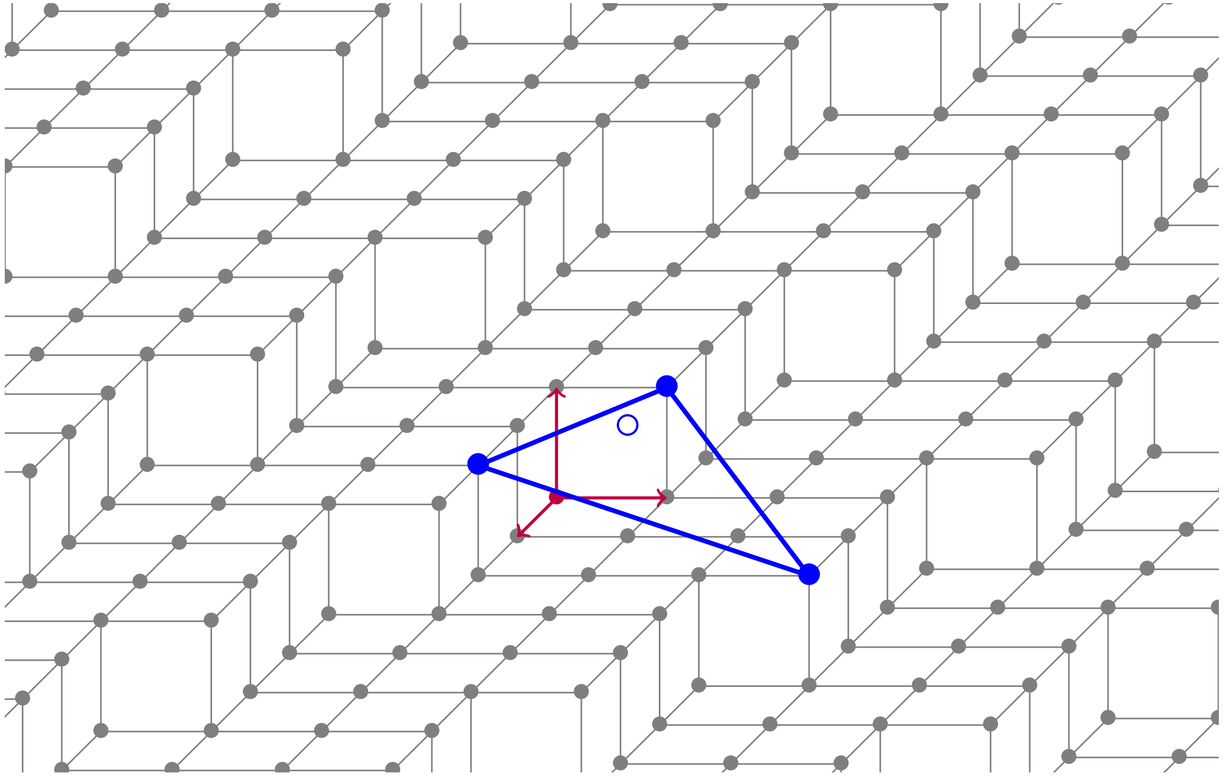

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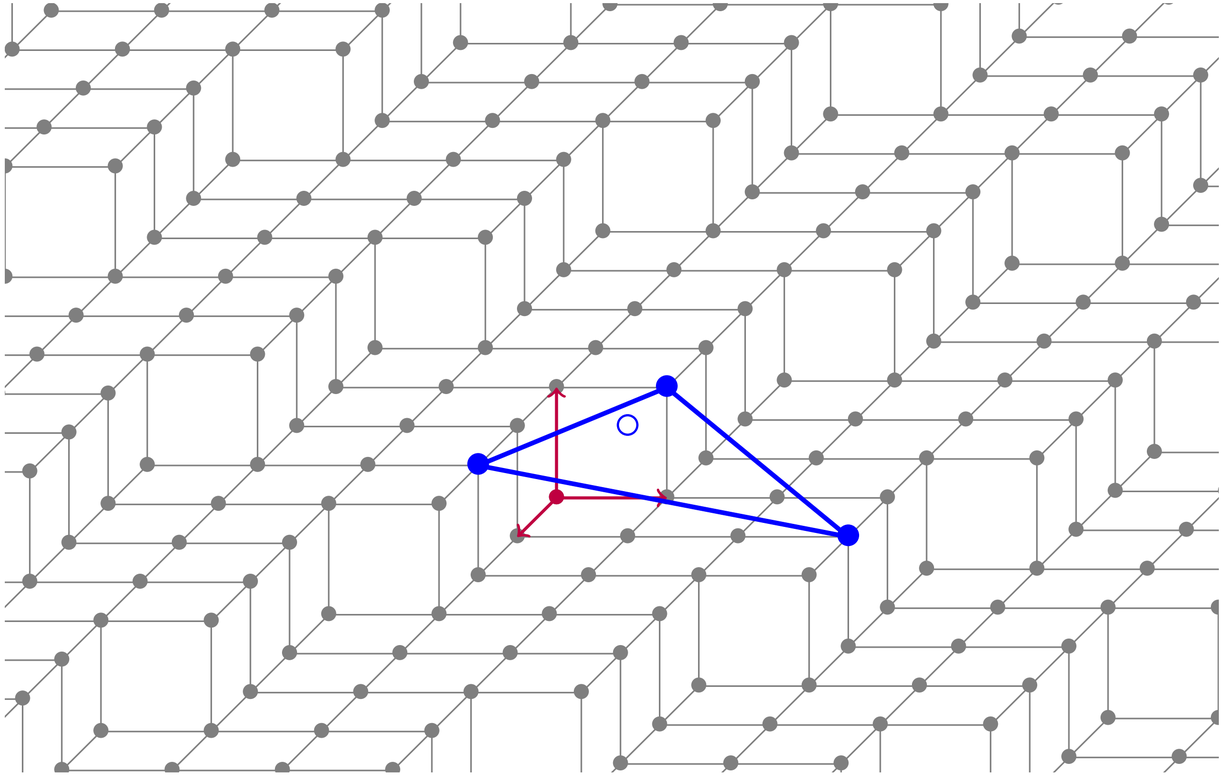

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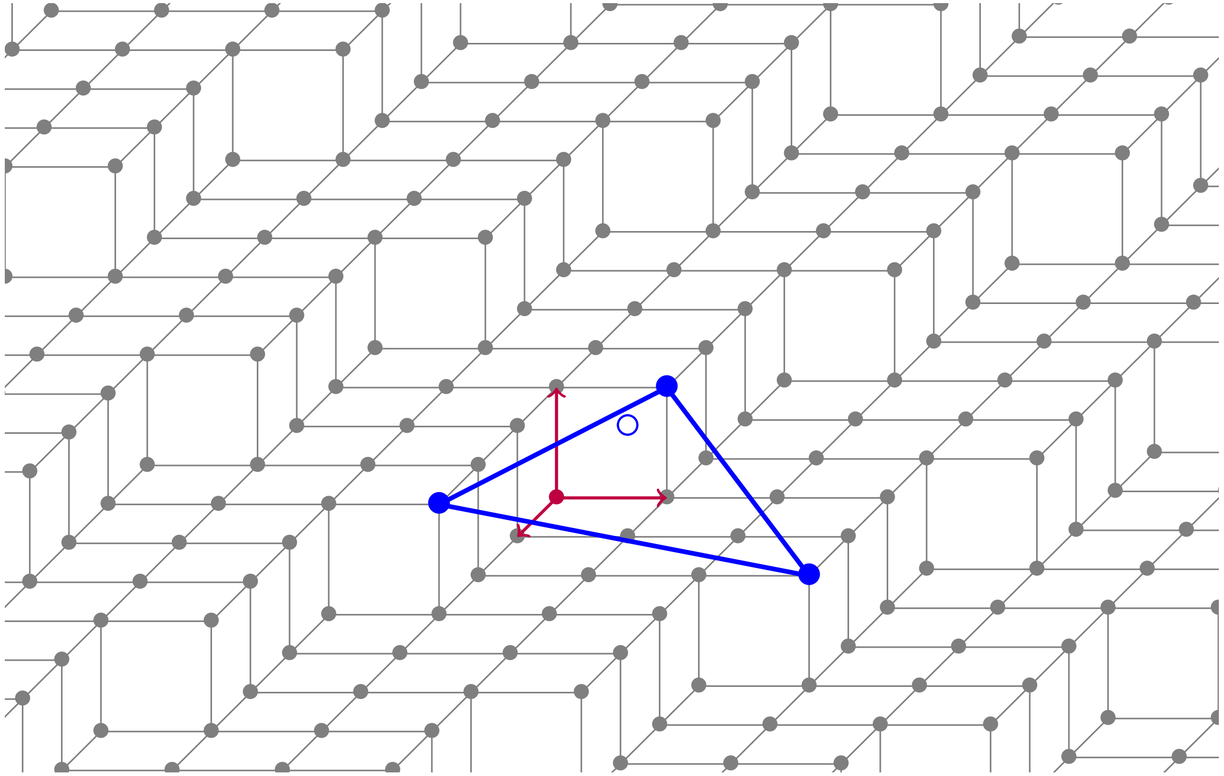

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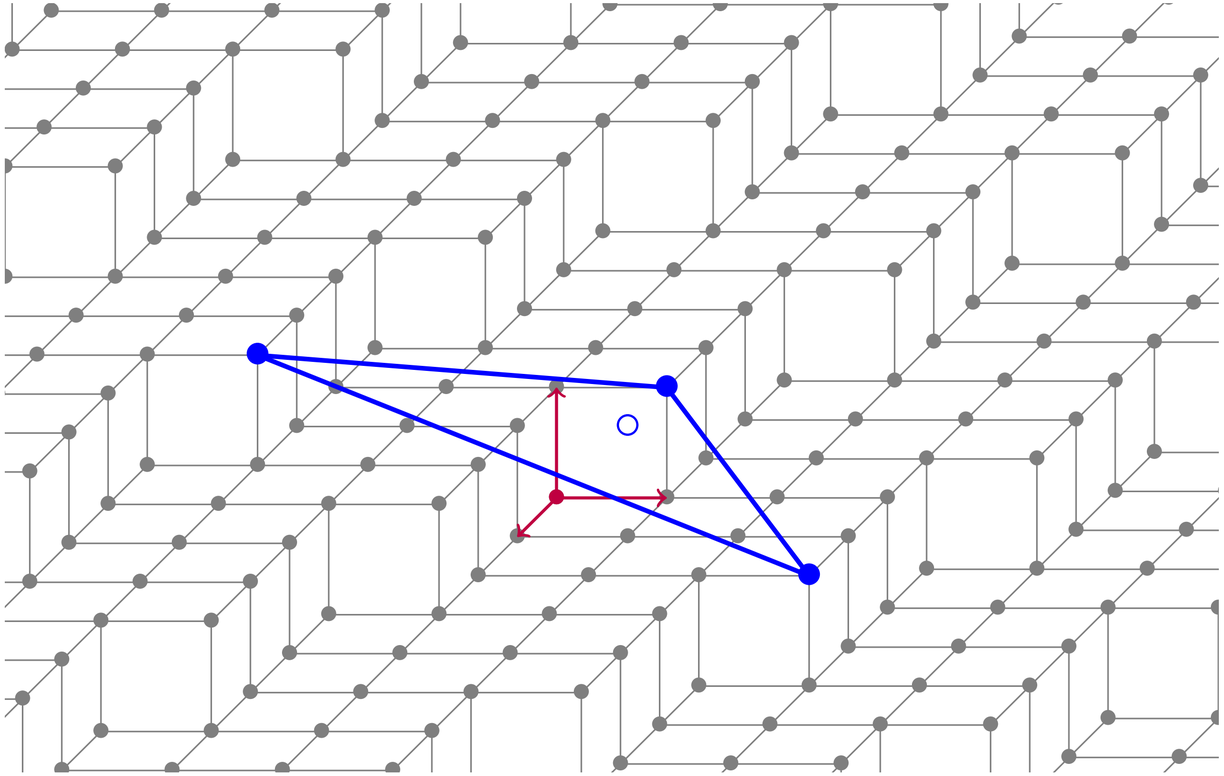

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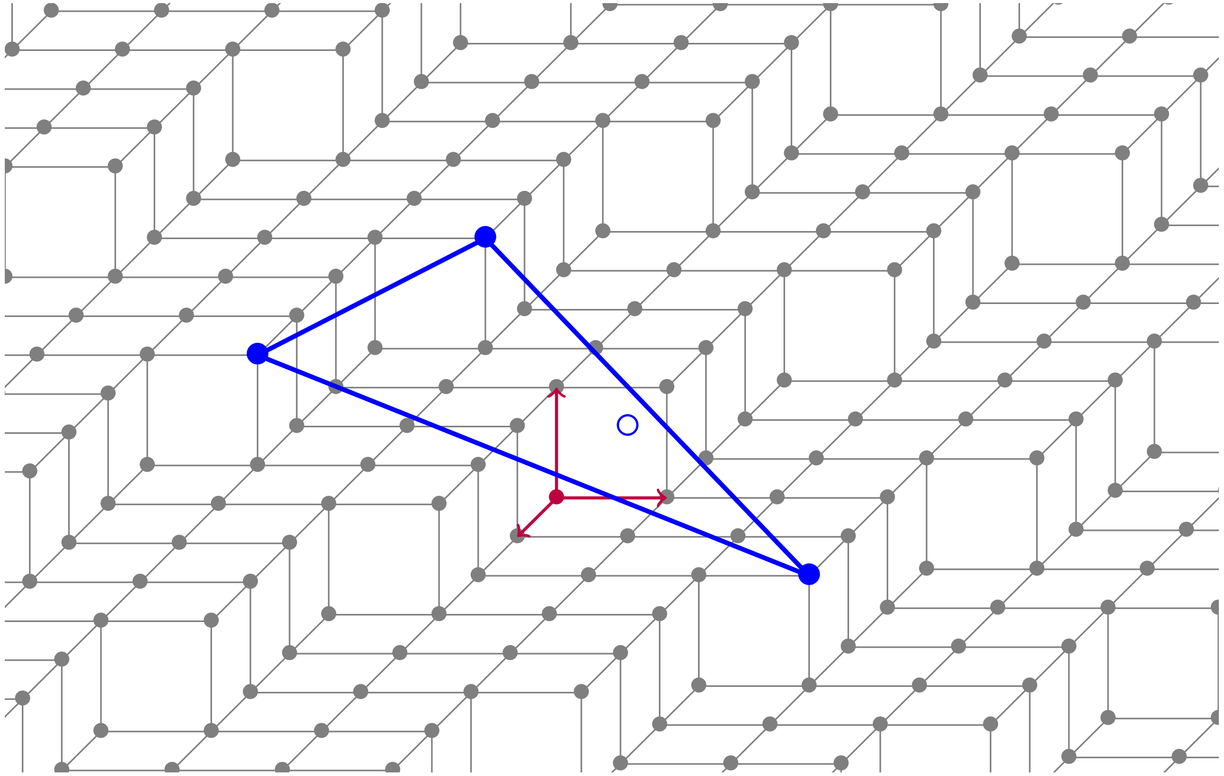

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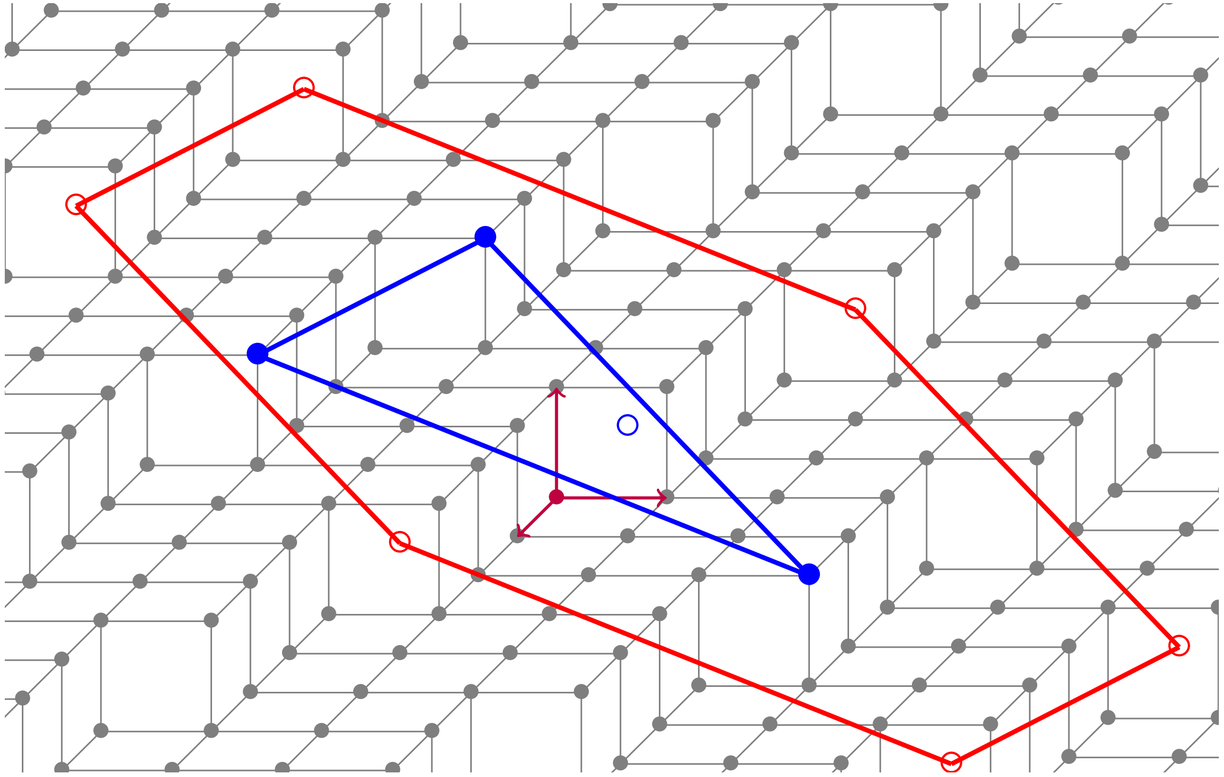
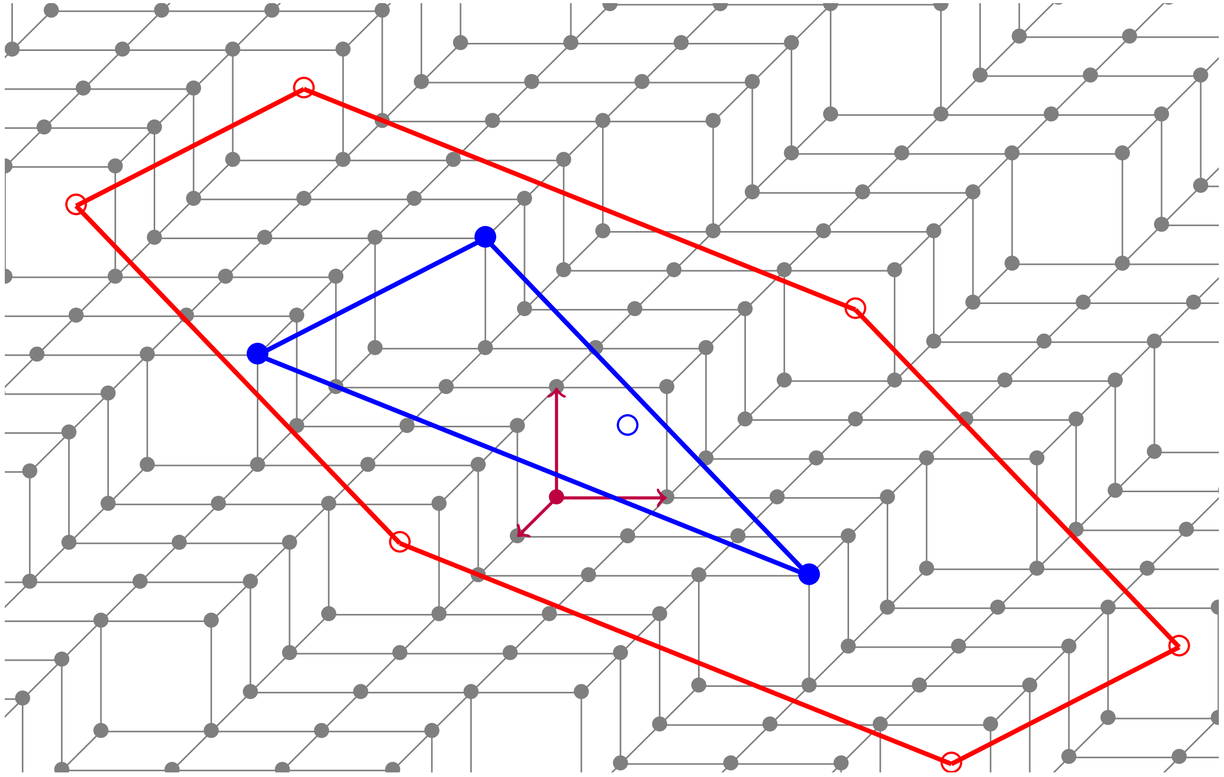


Digital geometry in 3d (parameter-free)

Local estimation of normal vector with successive sparse probings
[L., Provençal, Roussillon 15-24]

Normal estimation is **exact** on digital planes.
Worst-case complexity in $O(\omega \log \omega)$, average in $O(\log^2 \omega)$



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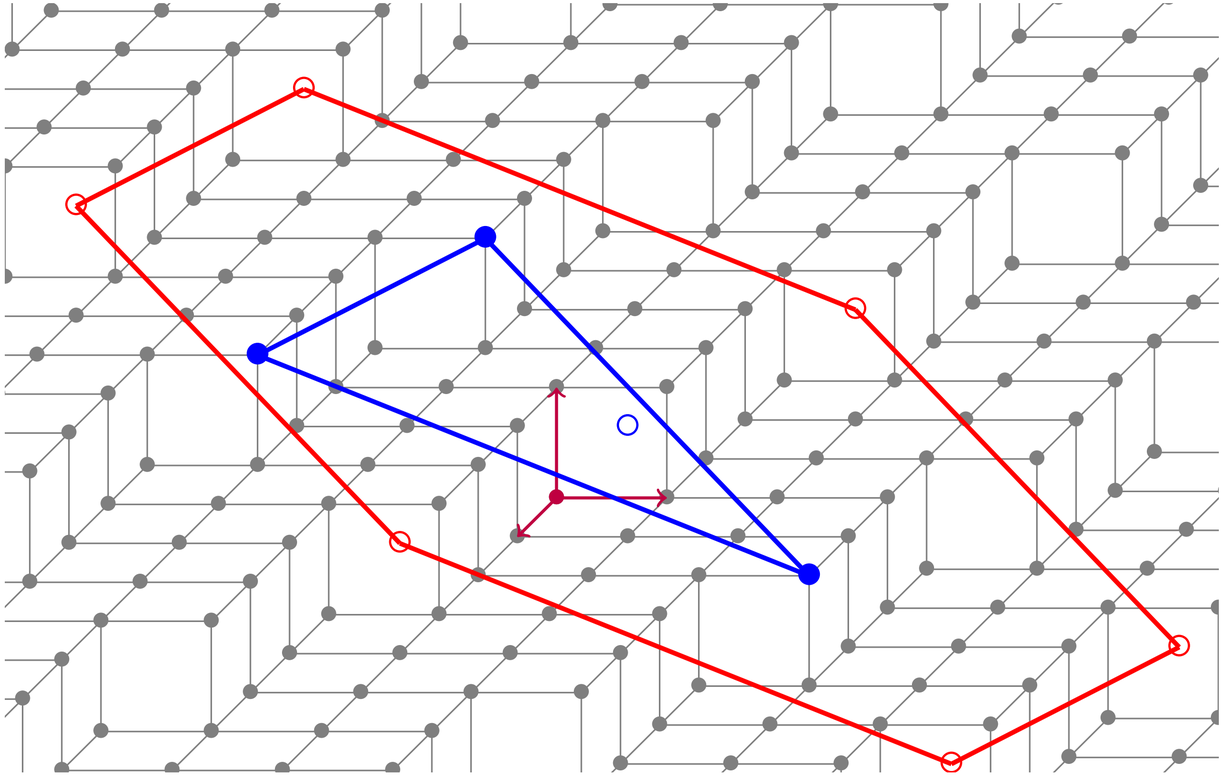
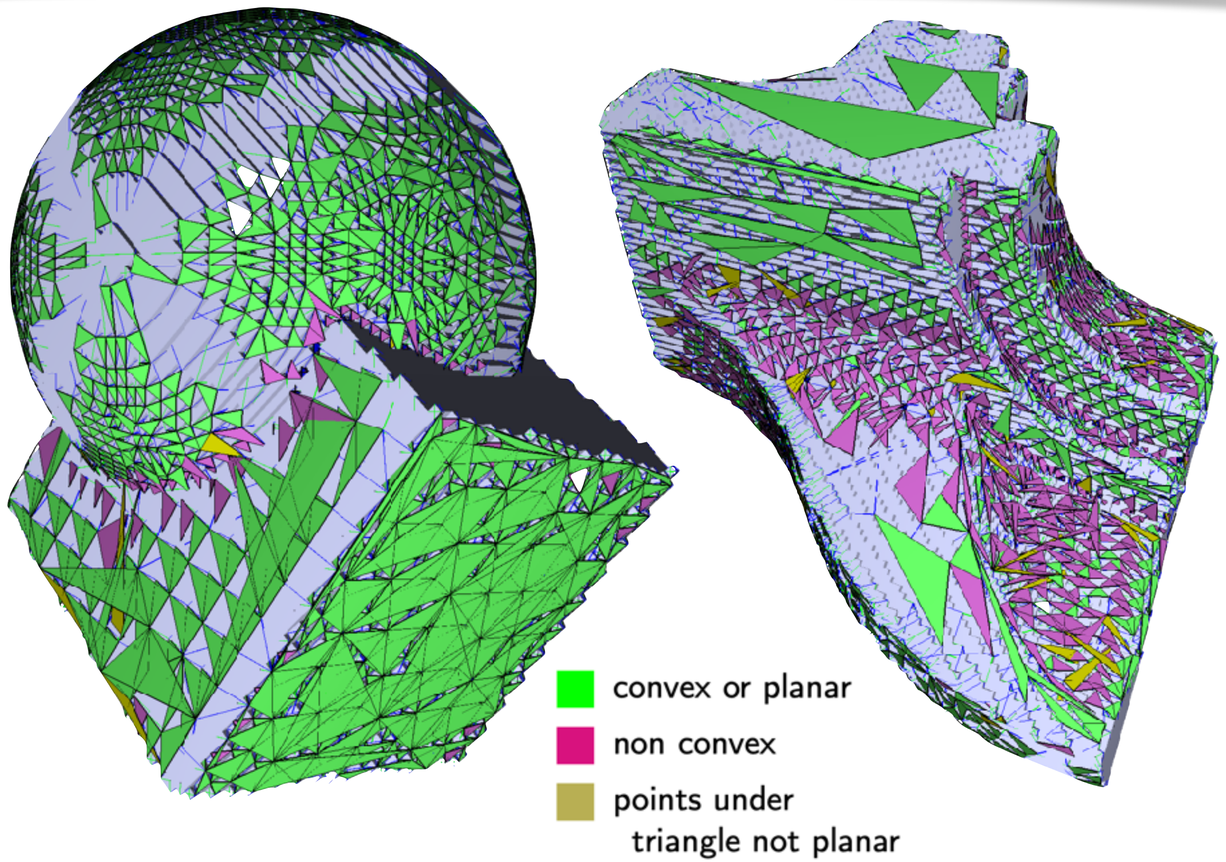


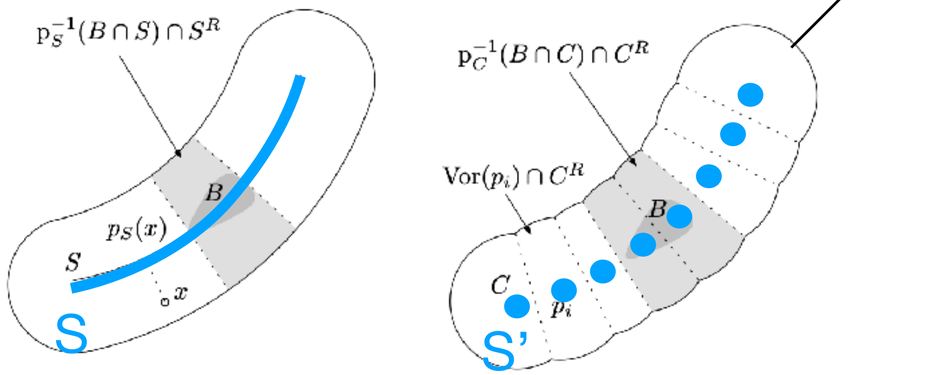


Voronoi Covariance Measures for estimating geometry

[Mérigot, Ovsjanikov, Guibas 10]

Voronoi Covariance Measure

$$V_{S,r}(B) = \int_{S^r \cap \pi_S^{-1}(B \cap S)} (\mathbf{x} - \pi_S(\mathbf{x})) \otimes (\mathbf{x} - \pi_S(\mathbf{x}))\, \mathrm{d}\mathbf{x}$$



For S and S’ two compact sets, then

$$\|V_{S,r}(B) - V_{S',r}\|_{\mathrm{op}} \leq O(\sqrt{d_H(S, S')})$$

Voronoi Covariance Measures for estimating geometry

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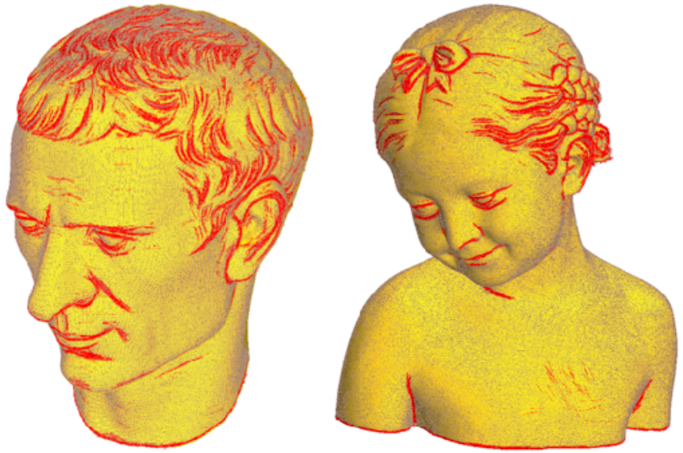
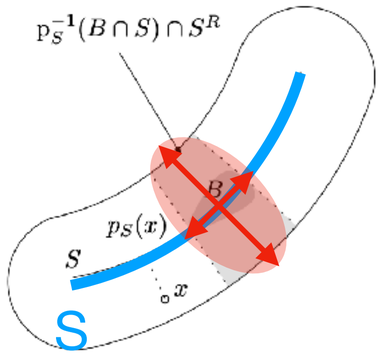
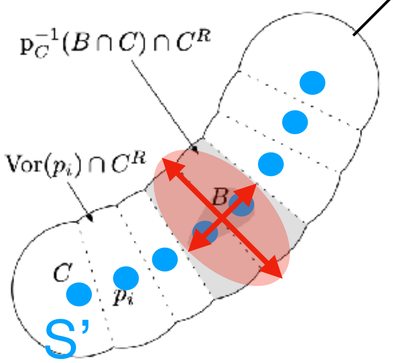






Eigendecomposition is stable
=> identifies normals+features

For S and S' two compact sets, then

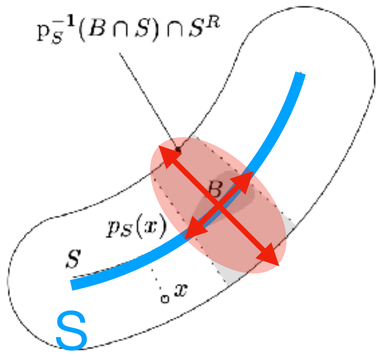
$$\|V_{S,r}(B) - V_{S',r}\|_{\mathrm{op}} \leq O(\sqrt{d_H(S, S')})$$

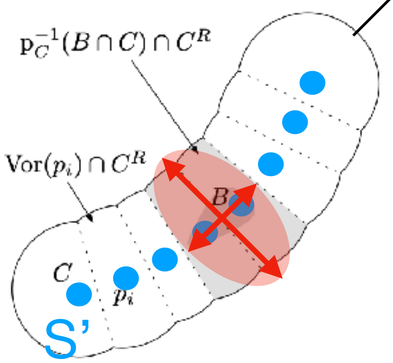
Voronoi Covariance Measures for estimating geometry

[Mérigot, Ovsjanikov, Guibas 10]

Voronoi Covariance Measure

$$V_{S,r}(B) = \int_{S^r \cap \pi_S^{-1}(B \cap S)} (\mathbf{x} - \pi_S(\mathbf{x})) \otimes (\mathbf{x} - \pi_S(\mathbf{x}))\, \mathrm{d}\mathbf{x}$$





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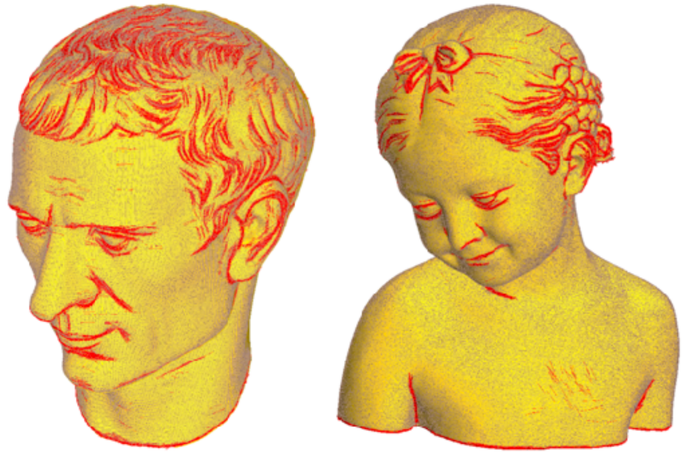

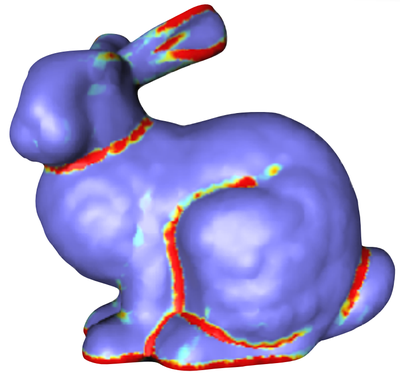

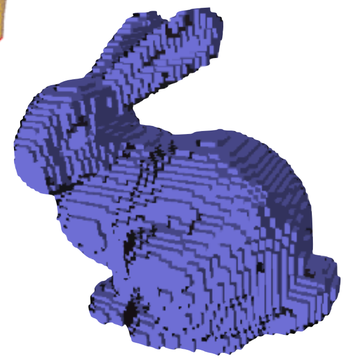

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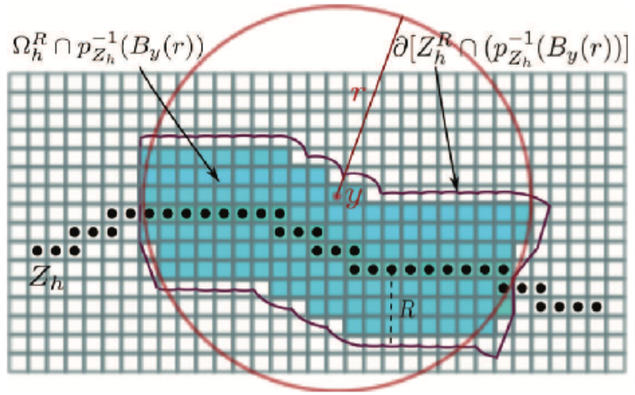
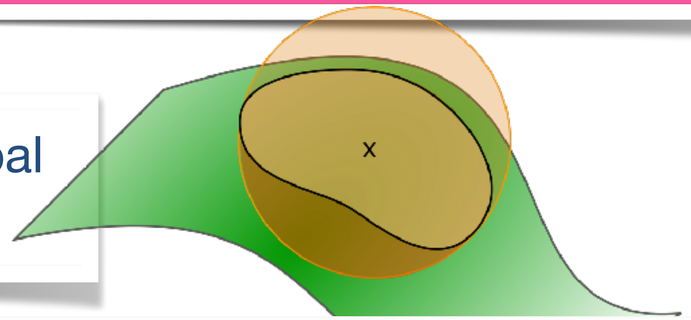
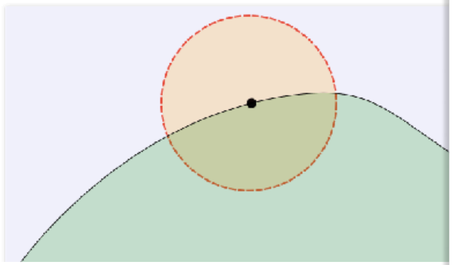


With B and r in $\Theta(h^{\frac{1}{4}})$, then first eigenvector is a convergent normal estimator with speed $O(h^{\frac{1}{8}})$
[Cuel, L., Thibert 14]

Integral Invariants for estimating geometry

Integral Invariants : analyzing $B_R(x) \cap X$ gives normals, principal directions and curvatures at x [Pottmann et al. 2007]





$$\int_{B_r(x)\cap X} \mathrm{d}x = \frac{\pi r^2}{2} - \frac{r^3}{3}\kappa + O(r^4)$$

κ : 2d curvature

$$\int_{B_r(x)\cap X} \mathrm{d}x = \frac{2\pi r^3}{3} - \frac{\pi r^4}{4}H + O(r^5)$$

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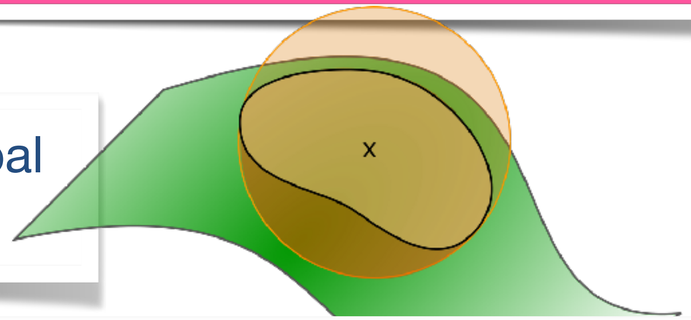

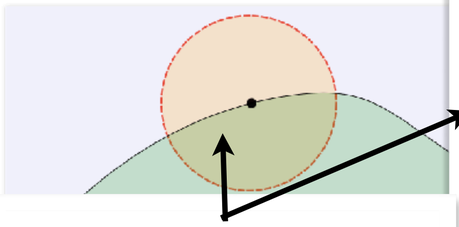

Area of $B_r(x) \cap X$

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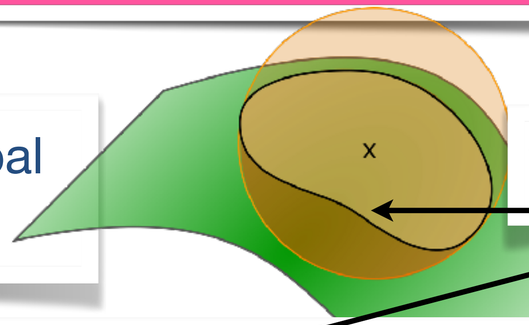
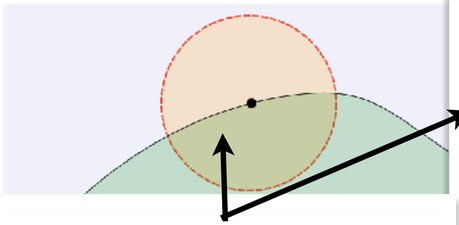


Volume of $B_r(x) \cap X$



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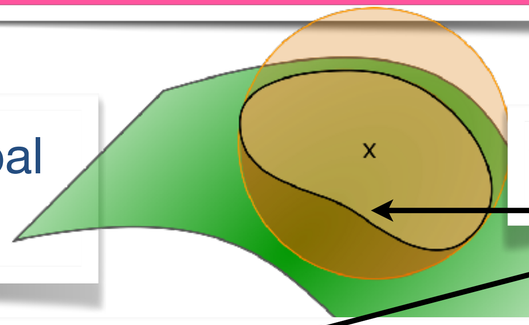

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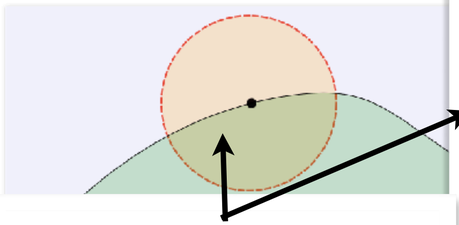

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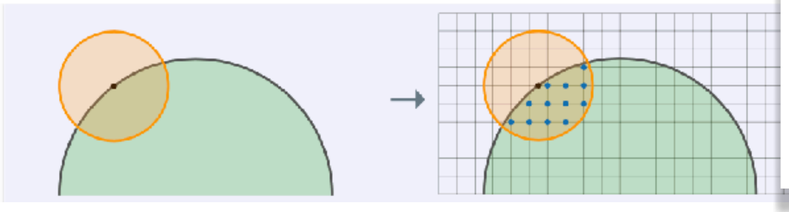

Area -> $\widehat{\text{Area}}$

Discretization $O(h/r^2)$

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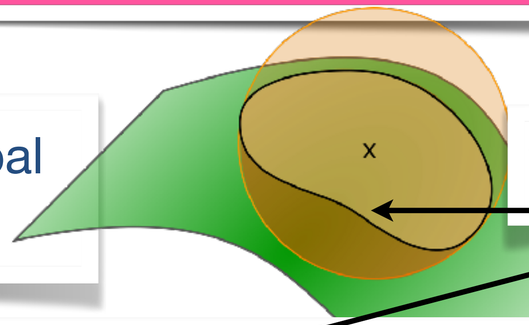

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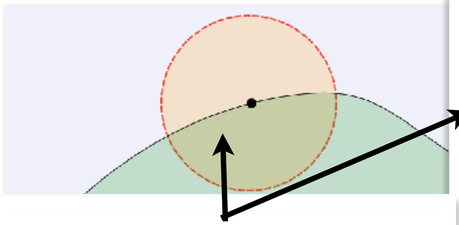

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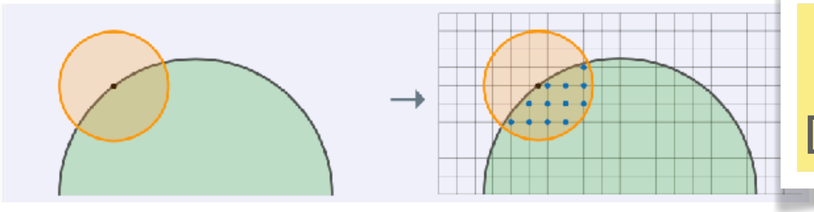

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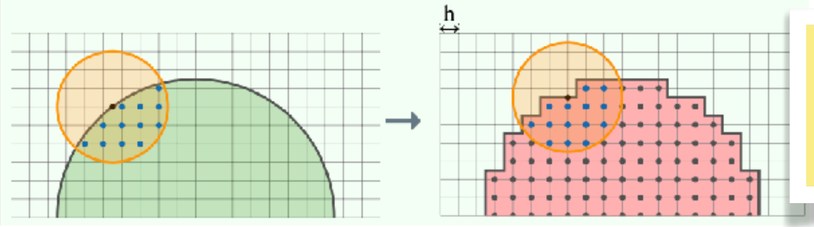
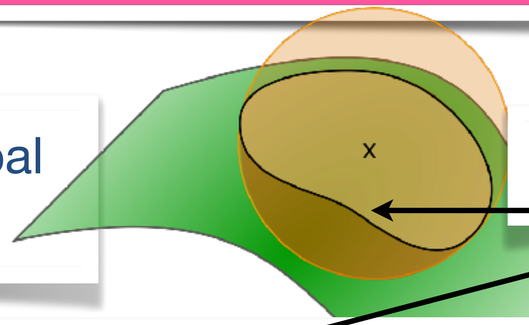


x -> $\hat{x}$

Position error $O(h/r^2)$

Integral Invariants for estimating geometry

Integral Invariants : analyzing $B_R(x) \cap X$ gives normals, principal directions and curvatures at x [Pottmann et al. 2007]



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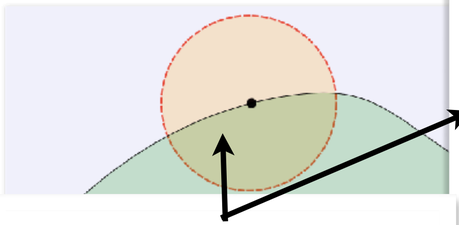

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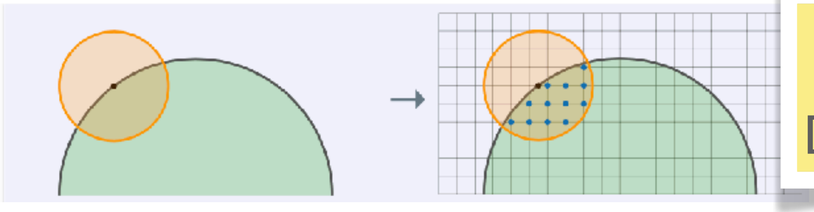

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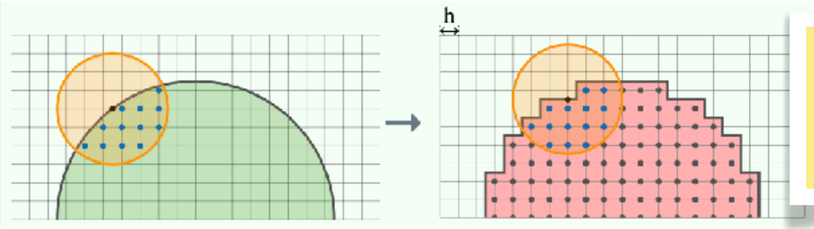


x -> $\hat{x}$

Position error $O(h/r^2)$

[Coeurjolly, L., Levallois 14]

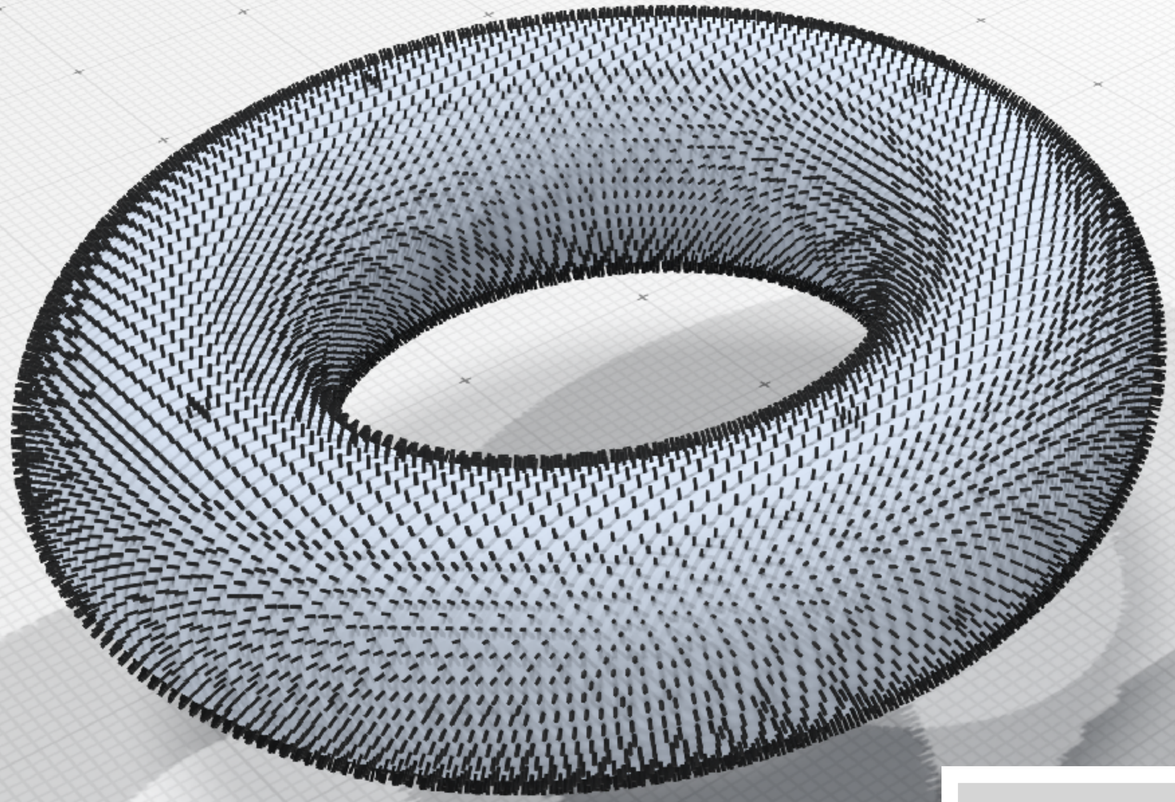
For **optimal radius** $r = O(h^{\frac{1}{3}})$,

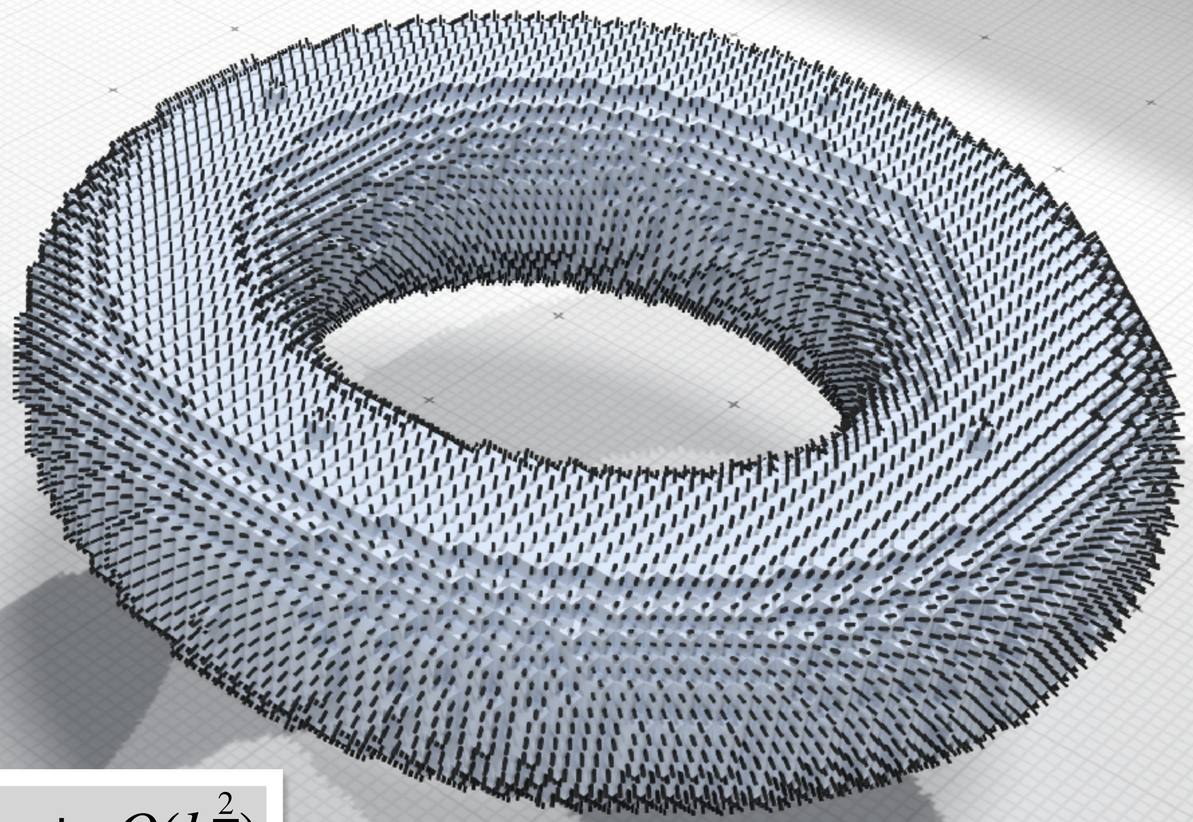
Curv. est. $\hat{\kappa} := \frac{3\pi}{2r} - \frac{3\,\widehat{\text{Area}}}{r^3}$, $\hat{H} := \frac{8}{3r} - \frac{4\,\widehat{\text{Vol}}}{\pi r^4}$

Both **converge in** $O(h^{\frac{1}{3}})$.

Using different **moments of** $B_r(x) \cap X$, we define normal, principal directions and curvatures estimators

Integral Invariants Normals are good

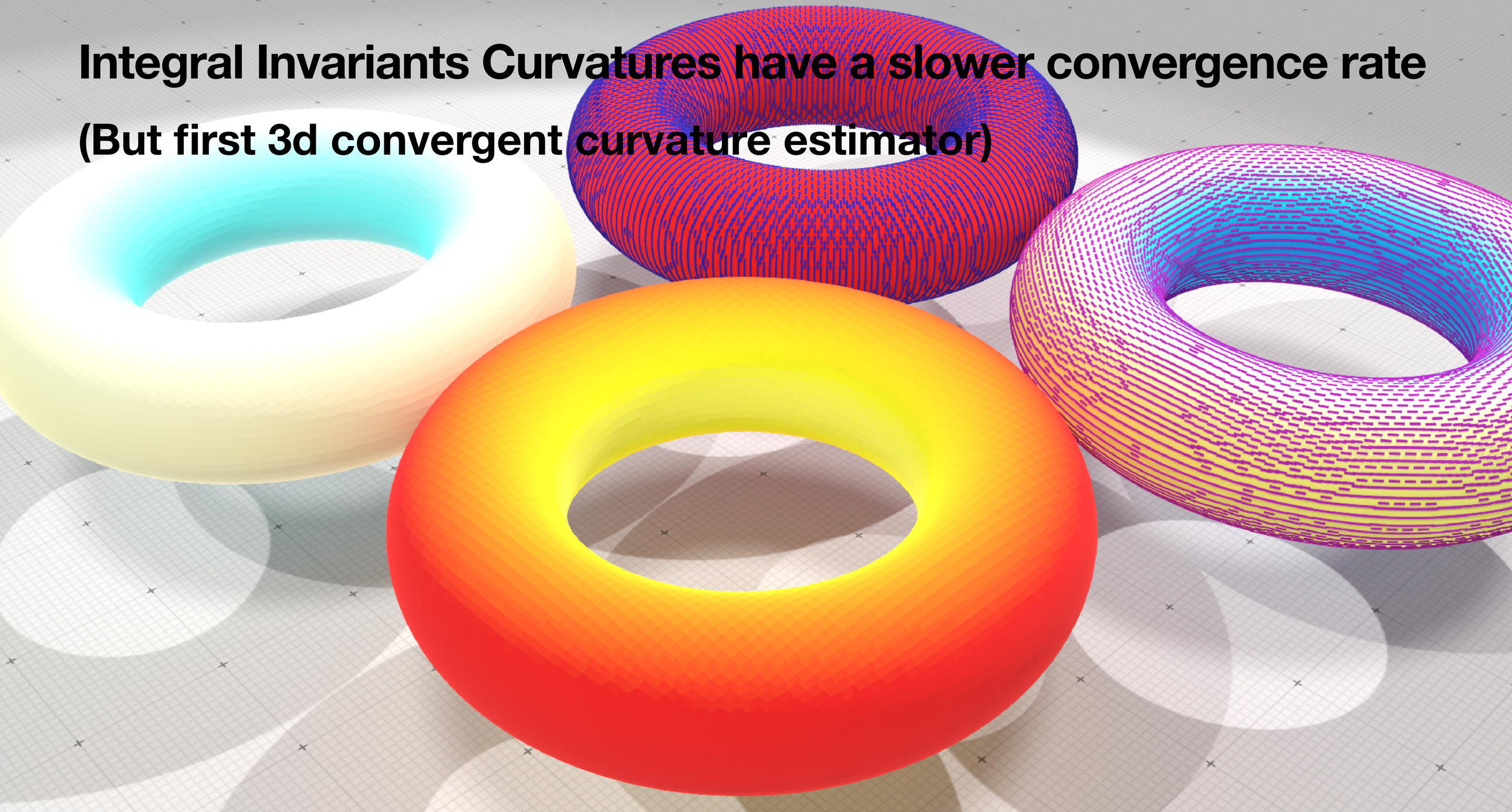




Convergence in $O(h^{\frac{2}{3}})$

Integral Invariants Curvatures have a slower convergence rate

(But first 3d convergent curvature estimator)



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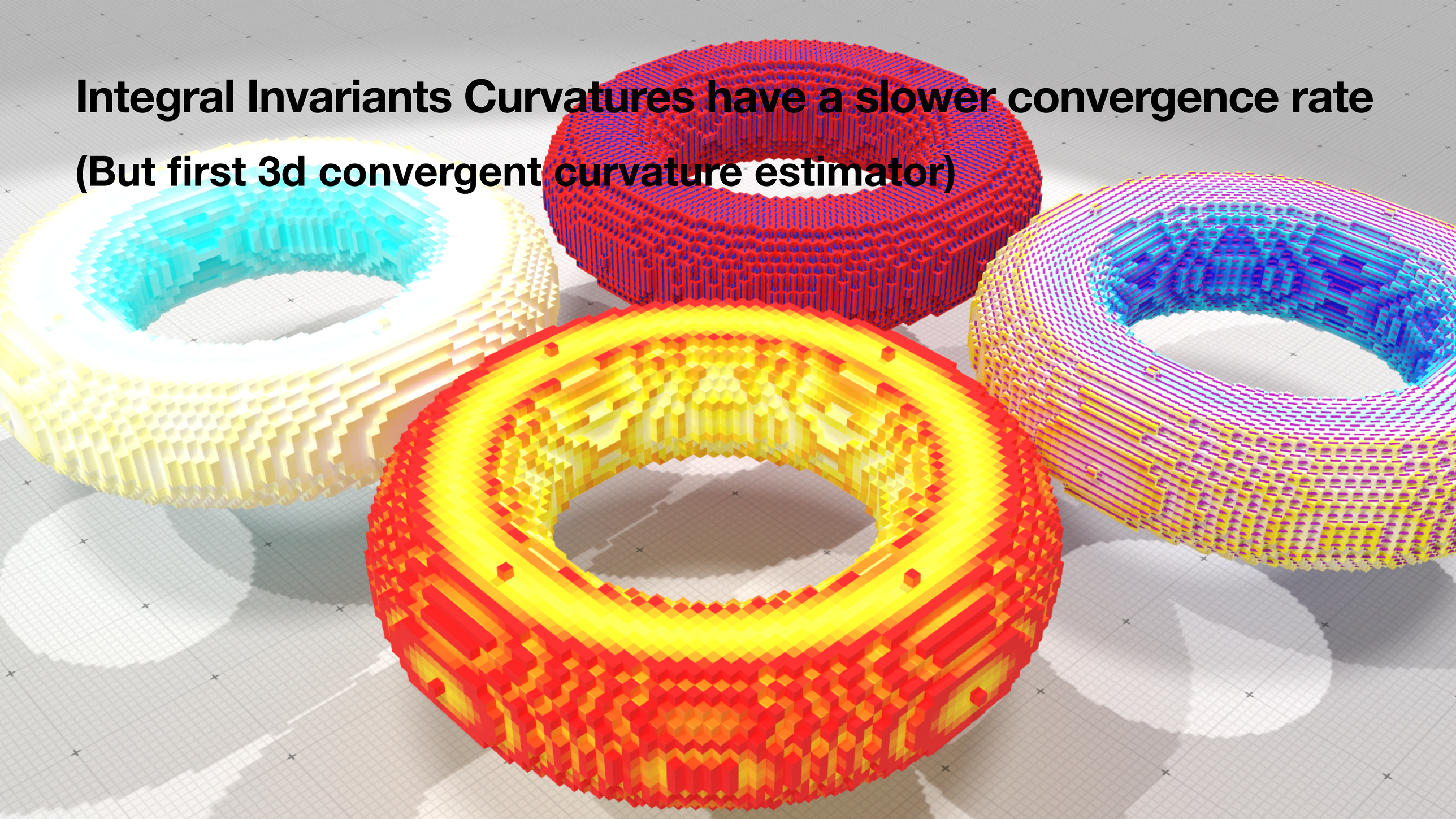

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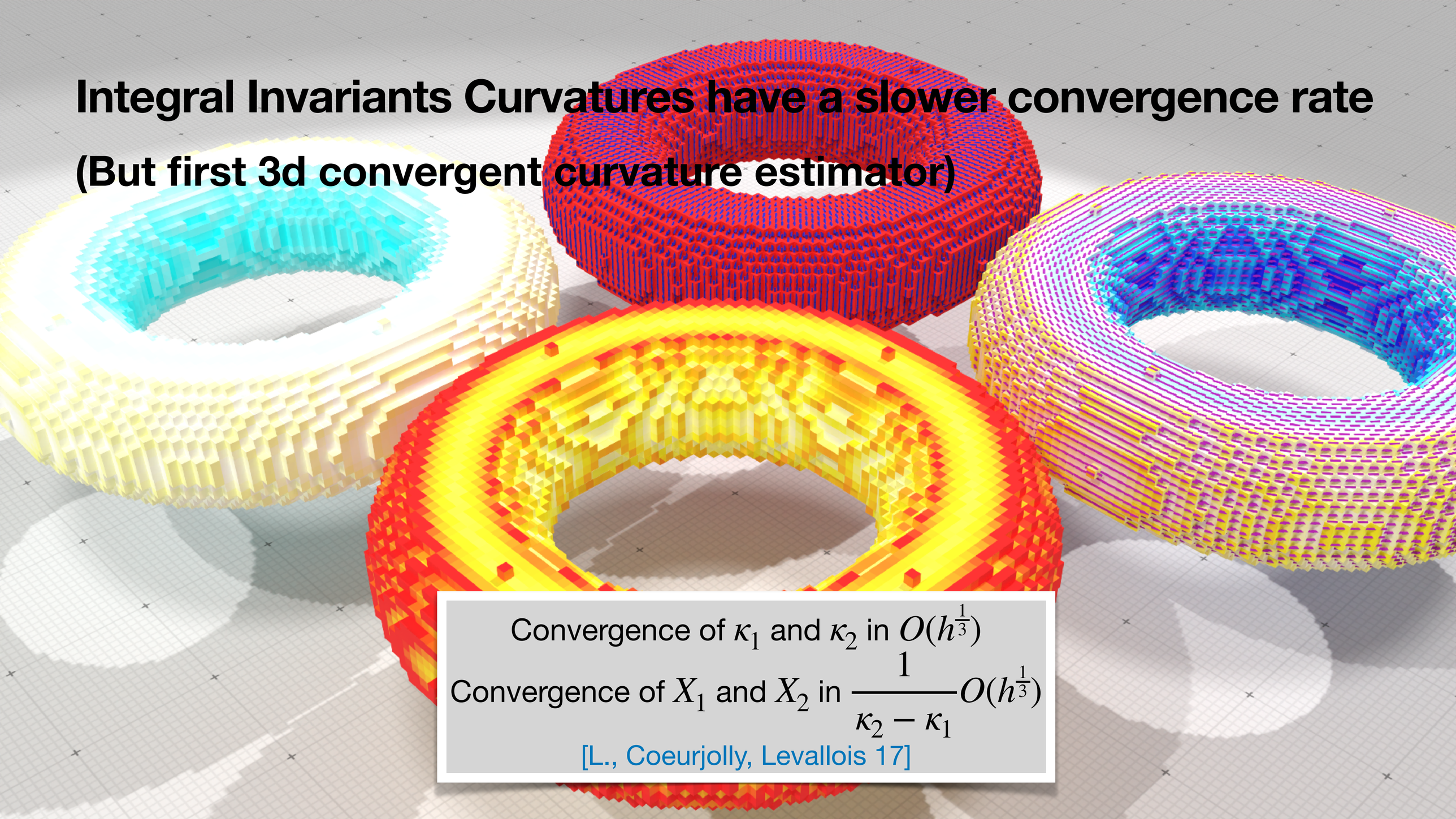
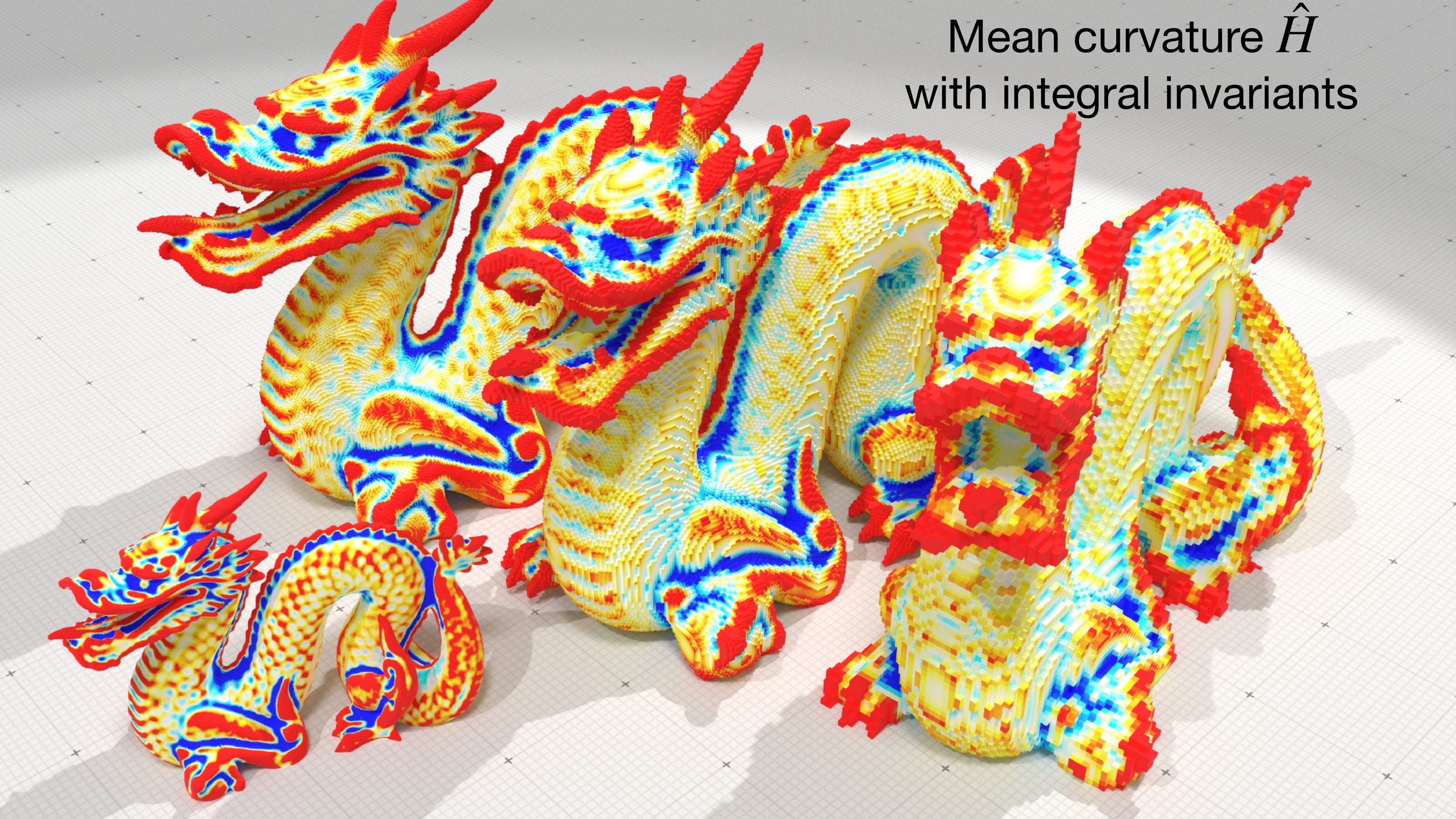


Convergence of κ_1 and κ_2 in $O(h^{\frac{1}{3}})$

Convergence of X_1 and X_2 in $\dfrac{1}{\kappa_2 - \kappa_1} O(h^{\frac{1}{3}})$

[L., Coeurjolly, Levallois 17]


Mean curvature $\hat{H}$
with integral invariants

Corrected Normal Currents

A unified approach to curvatures

$\mathbf{x}$: positions

$\mathbf{n}$: normal vectors

$\mathbf{u}$: vector field

$(\mathbf{x}, \mathbf{n})$

$(\mathbf{x}, \mathbf{u})$

Grassmannian $\mathbb{R}^3 \times \mathbb{S}^2$

Geometric 2d « surface »

Corrected Normal Currents

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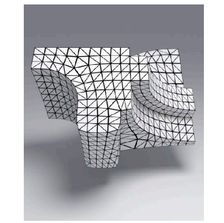
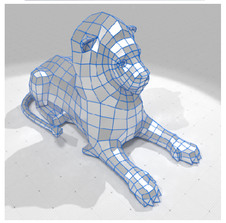

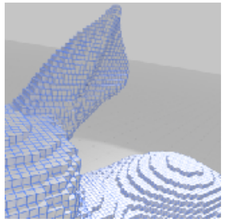

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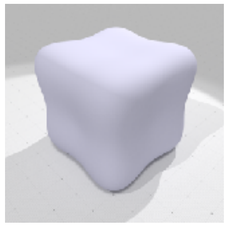

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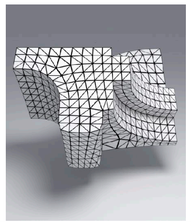
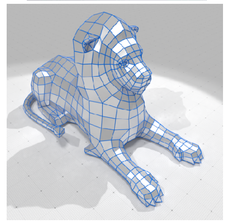

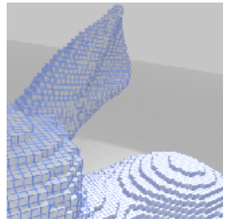

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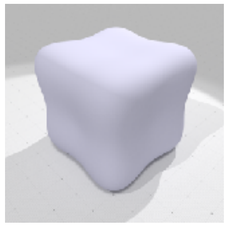


Lipschitz-Killing 2-forms

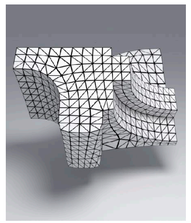
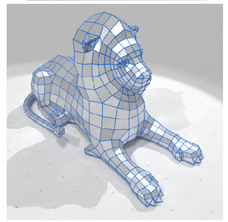
Grassmannian $\mathbb{R}^3 \times \mathbb{S}^2$

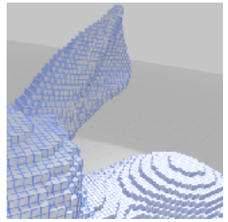
Geometric 2d « surface »

Curvature measures
(Mean, Gaussian, Area, anisotropic)

Corrected Normal Currents

A unified approach to curvatures



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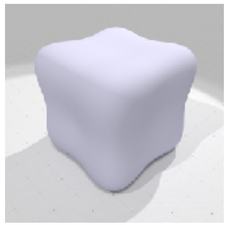

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Curvature measures
(Mean, Gaussian, Area, anisotropic)

« surfaces » are close
ϵ positions error
η normals error

Theorem (stability):
Curvatures measures are $\epsilon + \eta$ close

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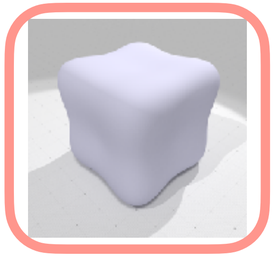
$(\mathbf{x}, \mathbf{n})$

$(\mathbf{x}, \mathbf{u})$



Normal cycle [Wintgen 82][Fu92] [Cohen-Steiner, Morvan 03 and 06]

Lipschitz-Killing 2-forms

Grassmannian $\mathbb{R}^3 \times \mathbb{S}^2$

Geometric 2d « surface »

Curvature measures (Mean, Gaussian, Area, anisotropic)

« surfaces » are close
ϵ positions error
η normals error

Theorem (stability): Curvatures measures are $\epsilon + \eta$ close

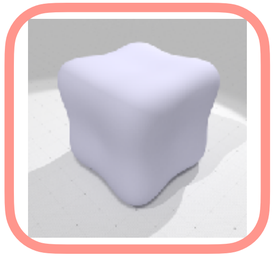
Corrected Normal Currents

A unified approach to curvatures

$\mathbf{x}$: positions
$\mathbf{n}$: normal vectors
$\mathbf{u}$: vector field

$(\mathbf{x}, \mathbf{n})$

$(\mathbf{x}, \mathbf{u})$



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Curvatures measures
are $\epsilon + \eta$ close

[L., Romon, Thibert 22]

Corrected Normal Currents

A unified approach to curvatures

Interpolated CNC measures for triangle $\tau=(\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k)$, normals $(\mathbf{u}_i, \mathbf{u}_j, \mathbf{u}_k)$

$\mathbb{R}^3 \times \mathbb{S}^2$

- *area measure:*

$$\mu_{\mathbf{u}}^{A}(\tau) = \int_{\tau} \Gamma_{\mathbf{u}}^{*}\omega^{A} = \frac{1}{2}\langle \bar{\mathbf{u}} \mid (\mathbf{x}_j - \mathbf{x}_i) \times (\mathbf{x}_k - \mathbf{x}_i) \rangle$$

- **m***ean curvature measure:*

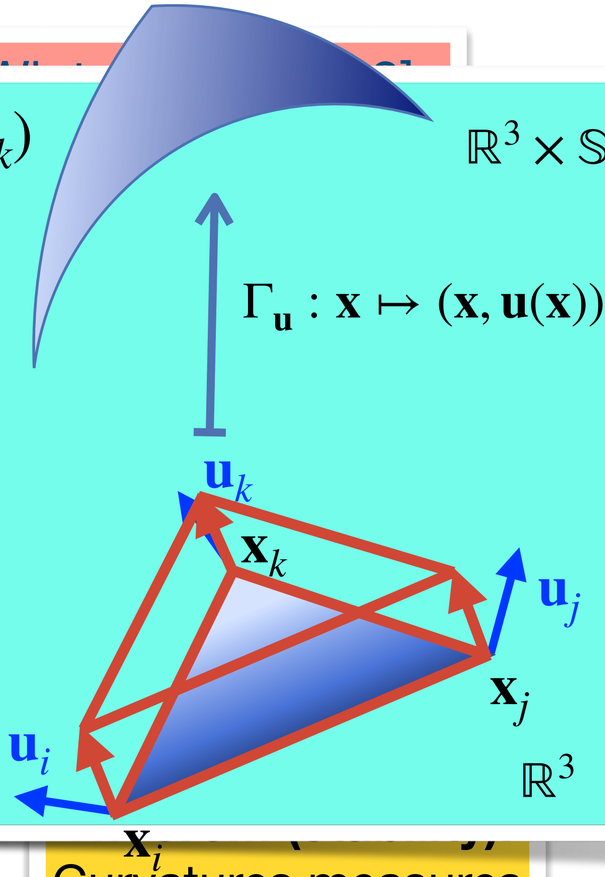
$$\mu_{\mathbf{u}}^{H}(\tau) = \int_{\tau} \Gamma_{\mathbf{u}}^{*}\omega^{H} = \frac{1}{2}\langle \bar{\mathbf{u}} \mid (\mathbf{u}_k - \mathbf{u}_j) \times \mathbf{x}_i + (\mathbf{u}_i - \mathbf{u}_k) \times \mathbf{x}_j + (\mathbf{u}_j - \mathbf{u}_i) \times \mathbf{x}_k \rangle$$

- *Gaussian curvature measure:*

$$\mu_{\mathbf{u}}^{G}(\tau) = \int_{\tau} \Gamma_{\mathbf{u}}^{*}\omega^{G} = \frac{1}{2}\langle \bar{\mathbf{u}} \mid (\mathbf{u}_j - \mathbf{u}_i) \times (\mathbf{u}_k - \mathbf{u}_i) \rangle$$

- *Anisotropic curvature tensor measure:*

$$\mu_{\mathbf{u}}^{(\mathbf{X},\mathbf{Y})}(\tau) = \frac{1}{2}\Big\langle \bar{\mathbf{u}} \mid \langle \mathbf{Y} \mid \mathbf{u}_k - \mathbf{u}_i \rangle \mathbf{X} \times (\mathbf{x}_j - \mathbf{x}_i) - \langle \mathbf{Y} \mid \mathbf{u}_j - \mathbf{u}_i \rangle \mathbf{X} \times (\mathbf{x}_k - \mathbf{x}_i) \Big\rangle$$



Simple formulas, linear in $\mathbb{R}^3 \times \mathbb{S}^2$

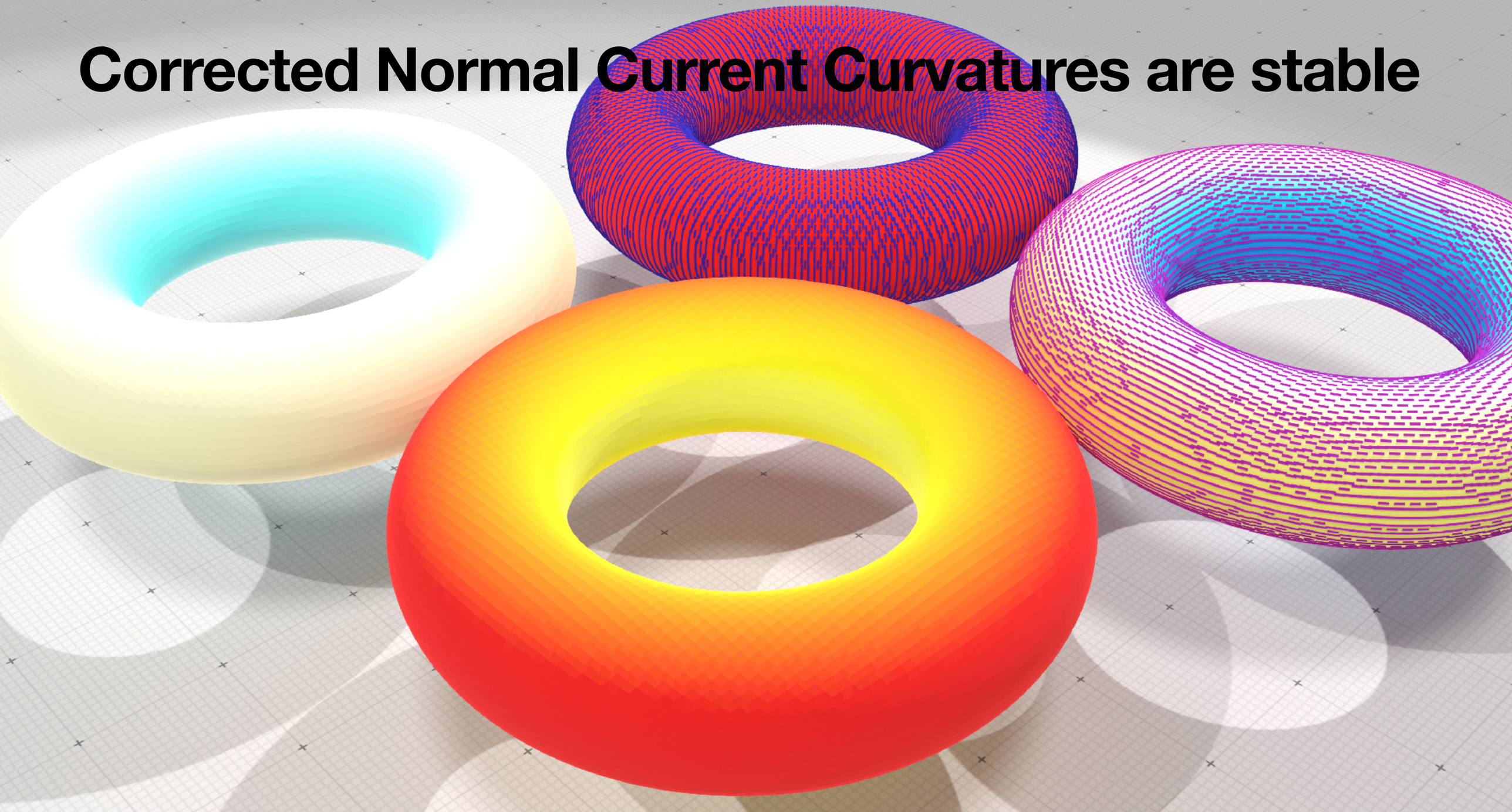
η normals error

Curvatures measures are $\epsilon + \eta$ close

$(\mathbf{x}, \mathbf{u})$

[L., Romon, Thibert 22]

Corrected Normal Current Curvatures are stable



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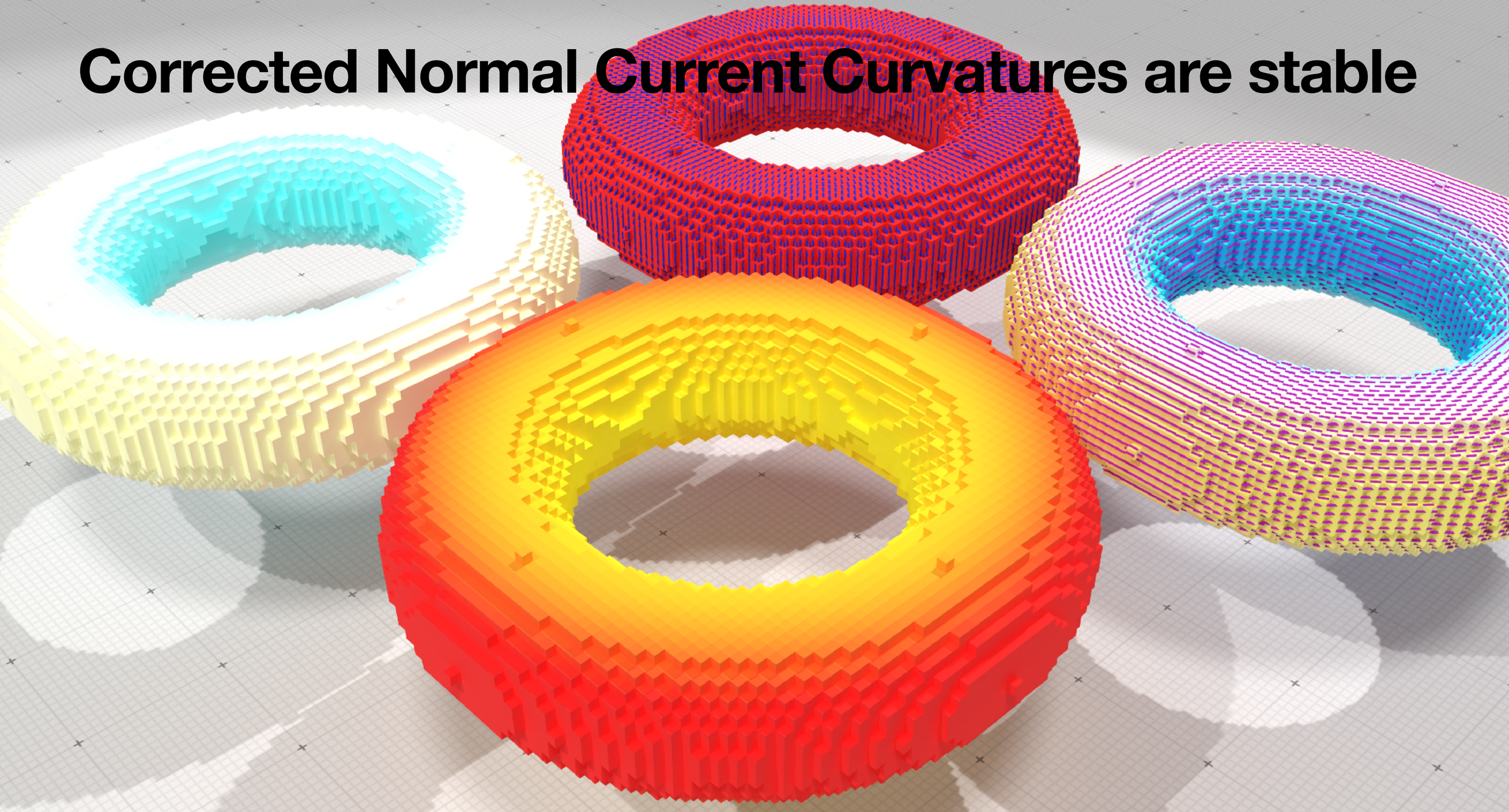

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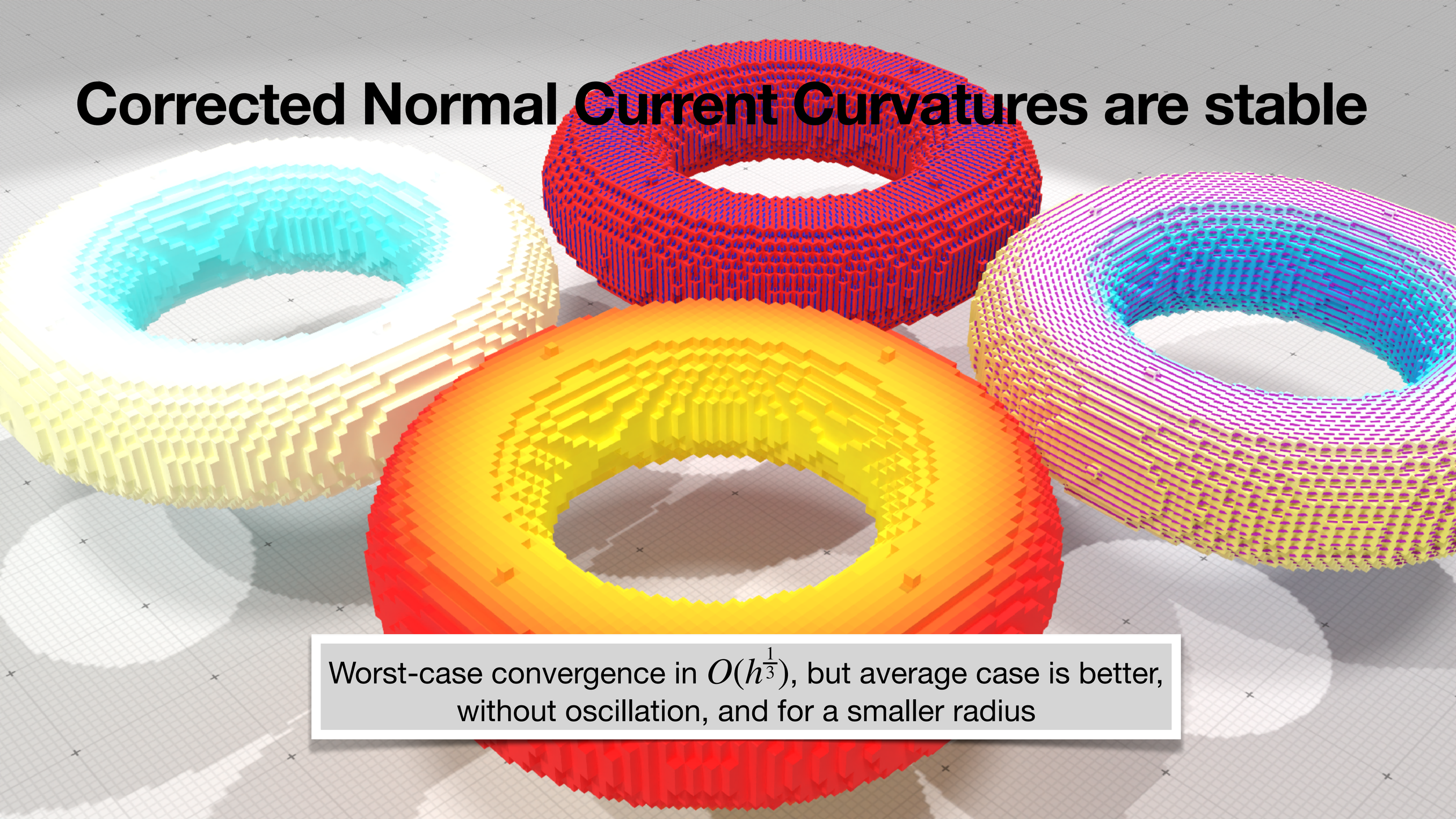
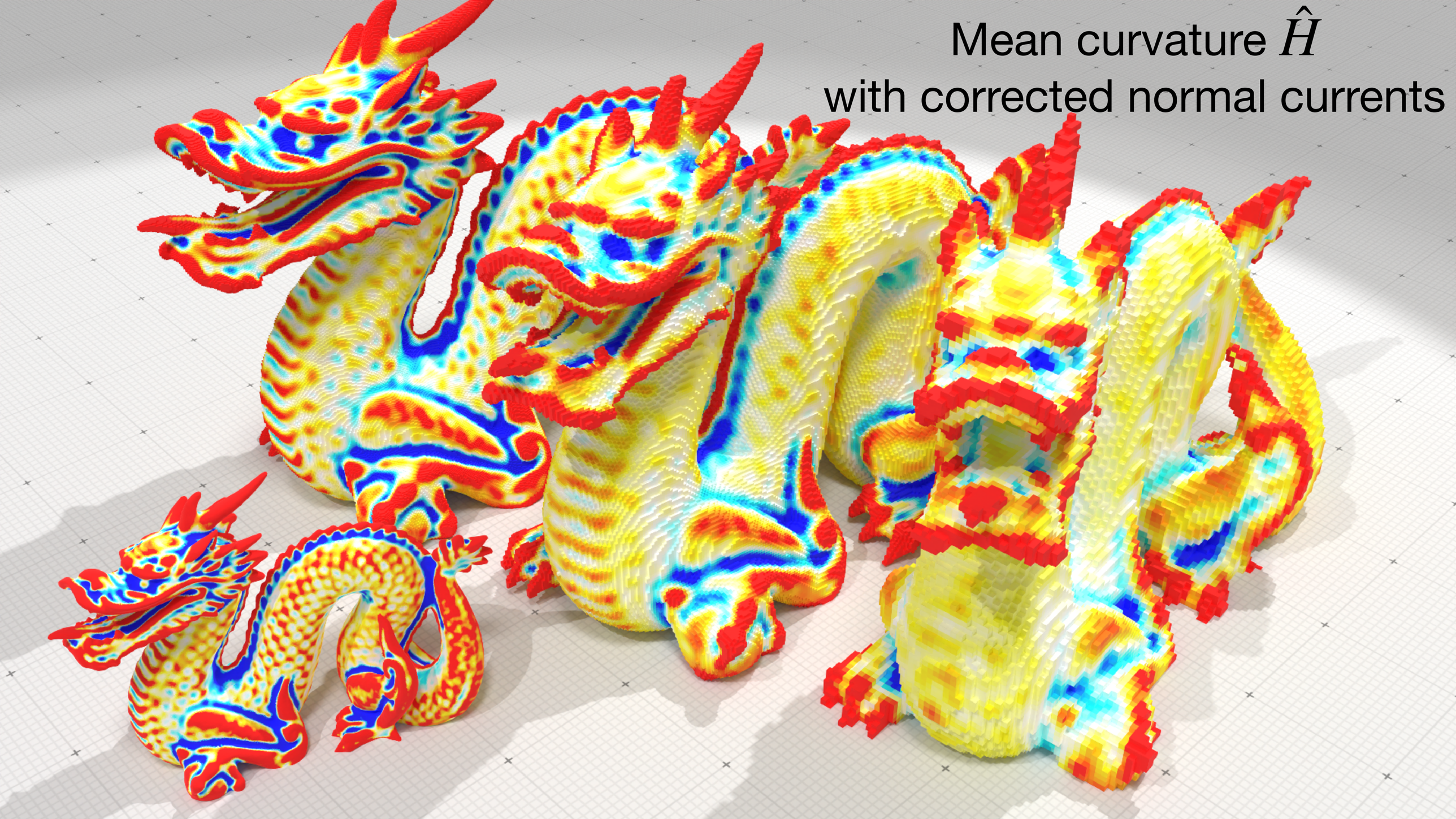
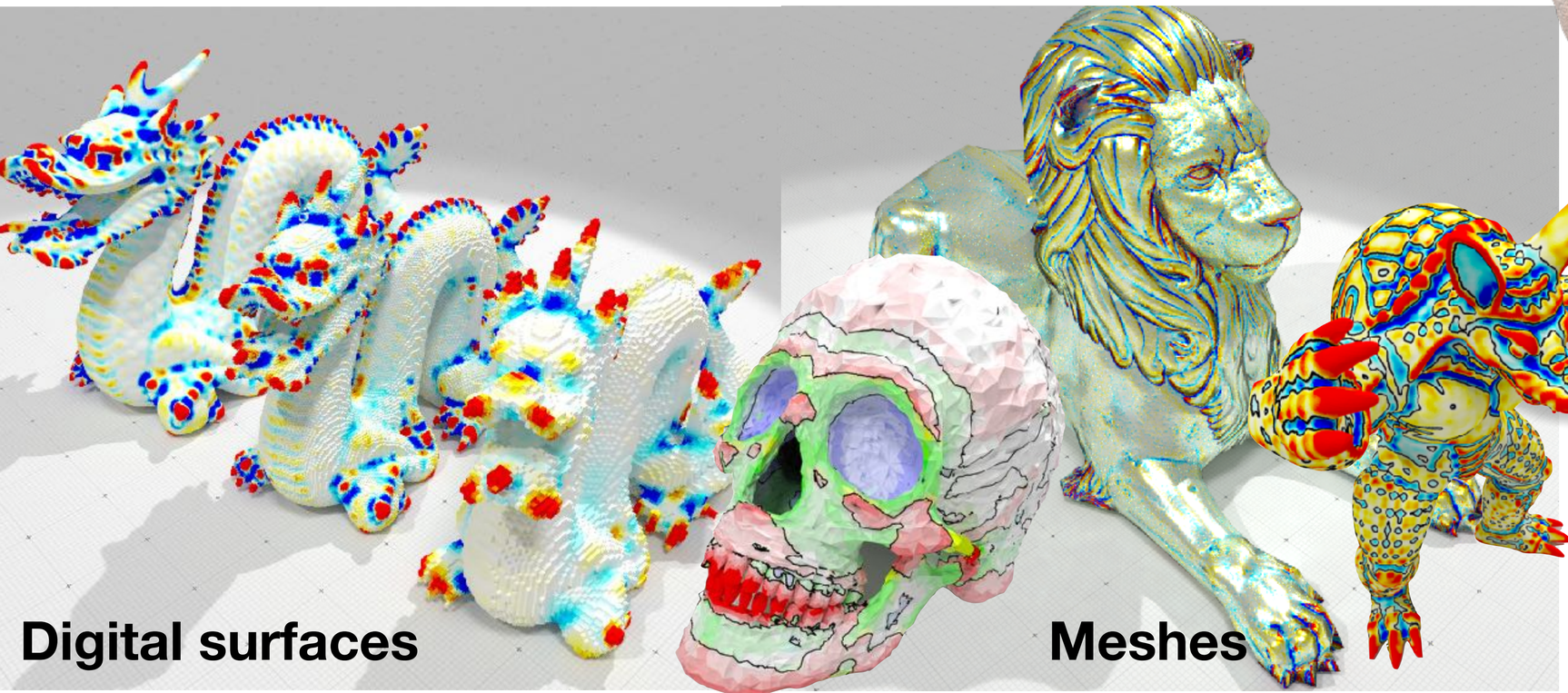


Worst-case convergence in $O(h^{\frac{1}{3}})$, but average case is better, without oscillation, and for a smaller radius


Mean curvature $\hat{H}$
with corrected normal currents

Corrected Normal Current

Interpolated CNC achieves state-of-the-art
- digital surfaces [L., Romon, Thibert 22]
- meshes [L., Coeurjolly, Romon, Thibert 21]
- point clouds [L., Coeurjolly, Labart, Romon, Thibert 23]



Digital surfaces

Meshes

Point clouds

Corrected Normal Current

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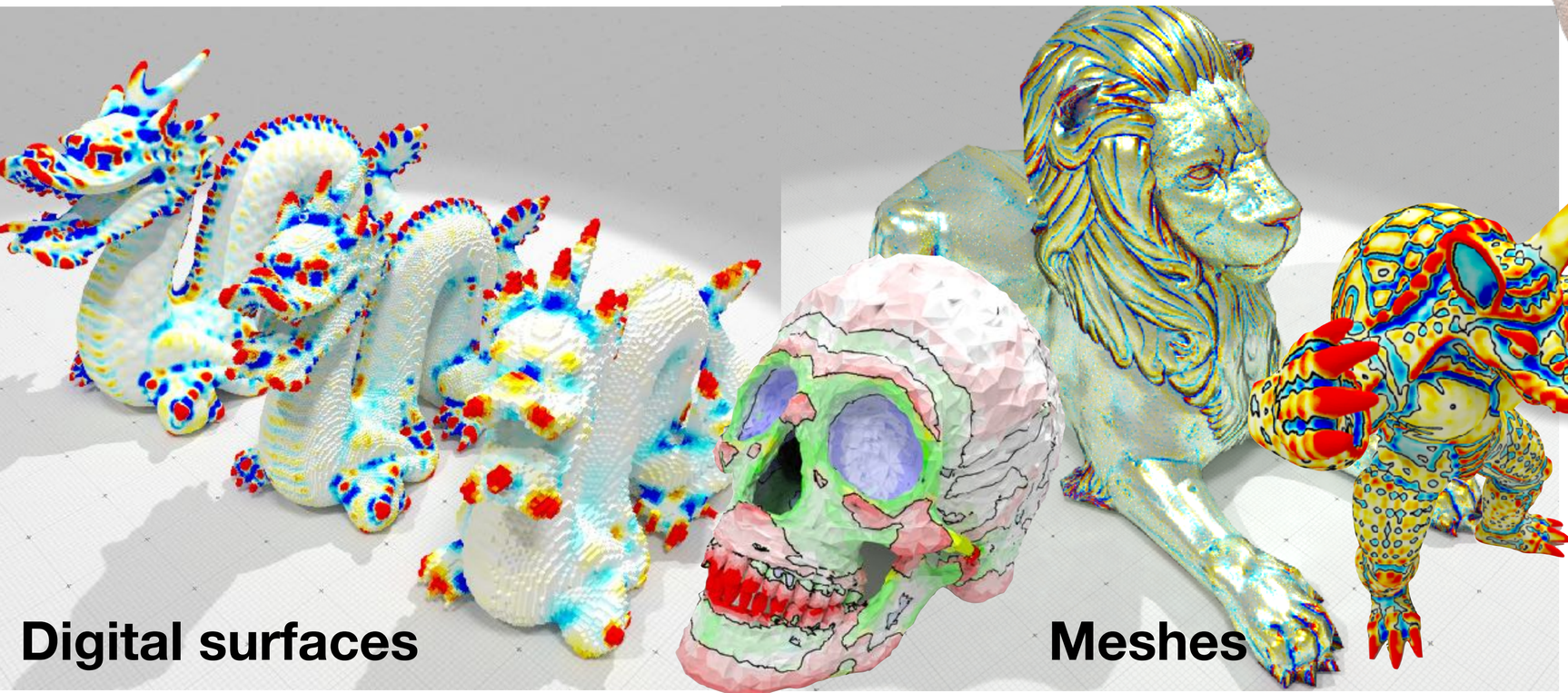

Digital surfaces

Meshes

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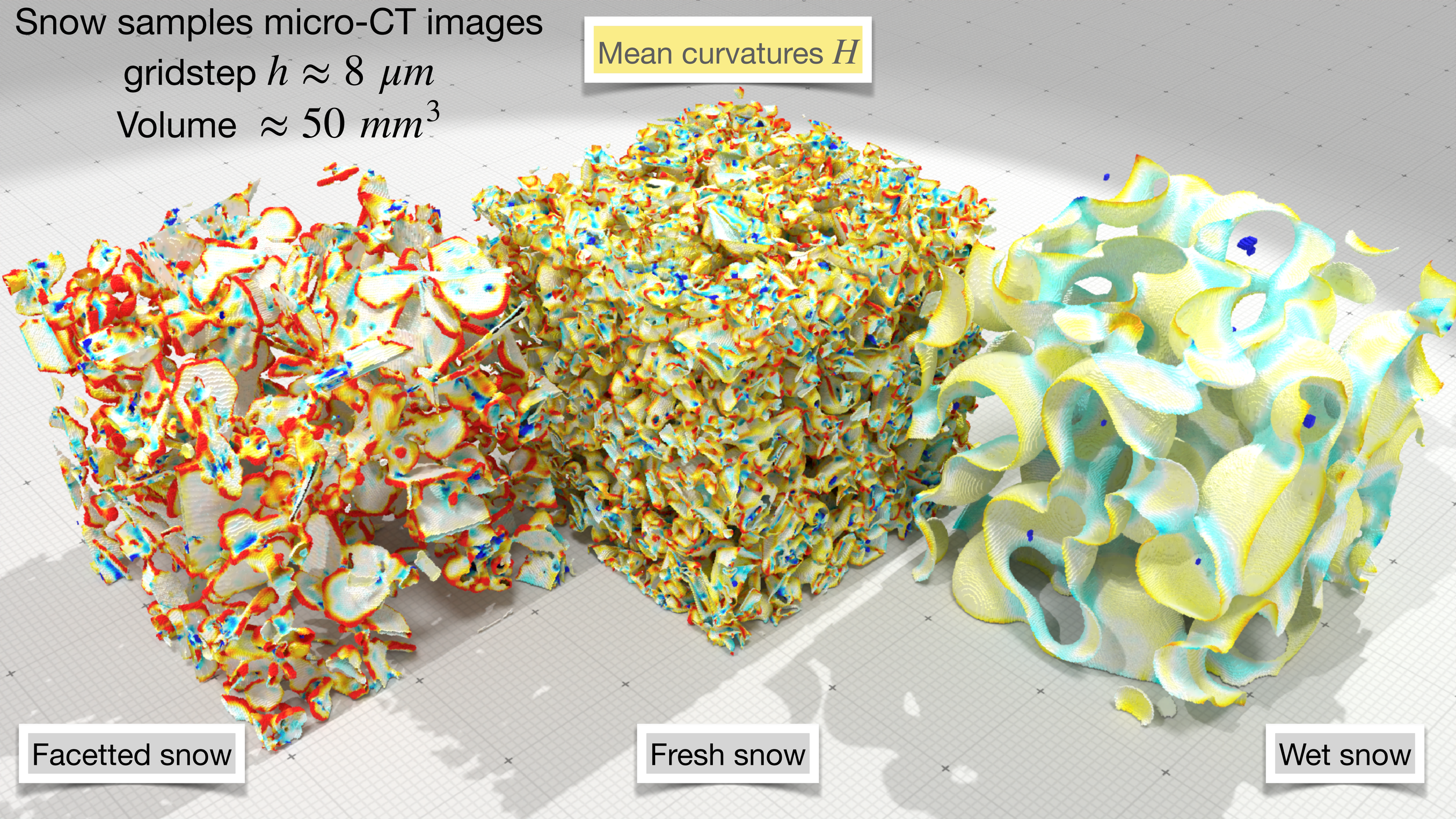
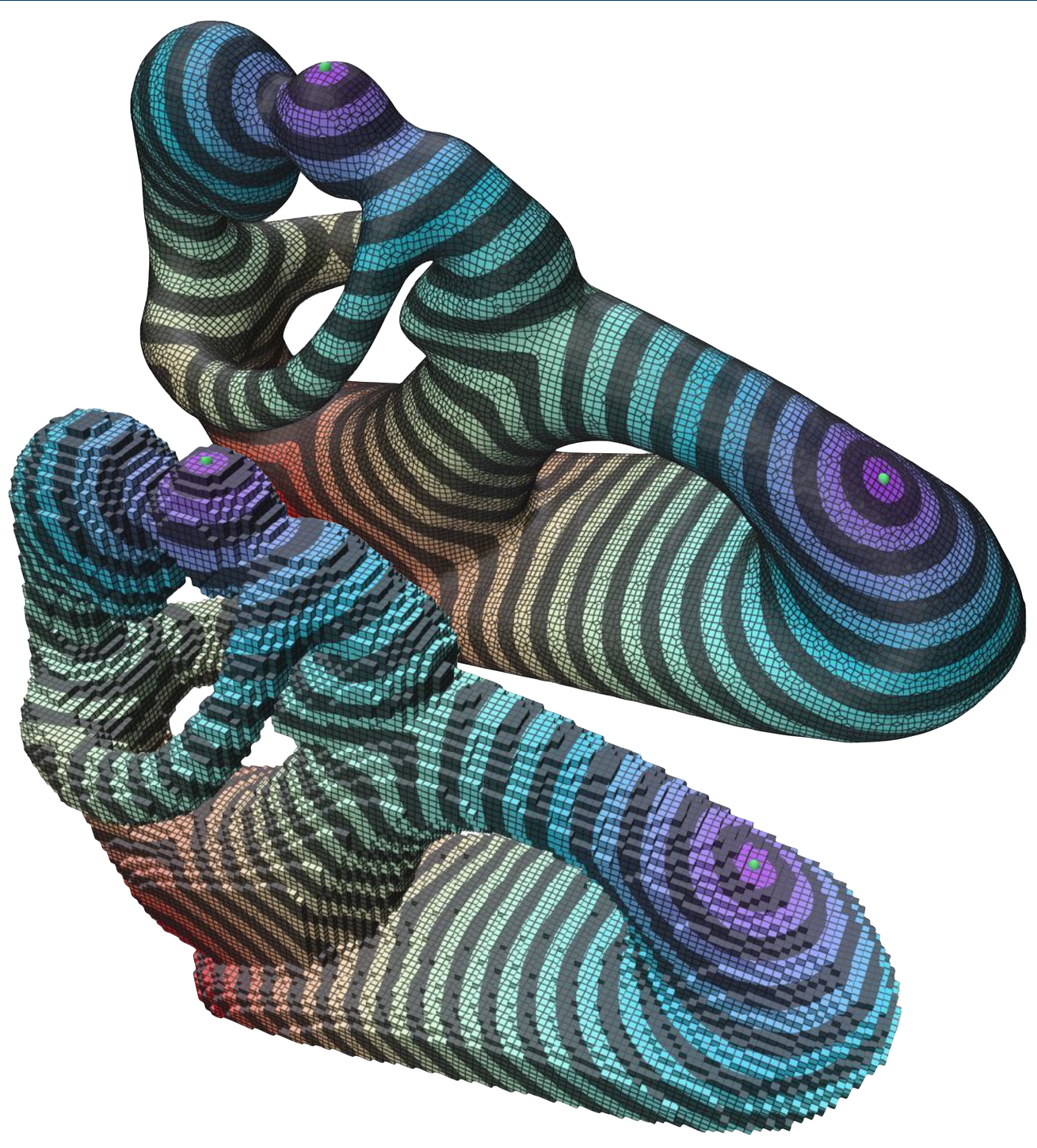
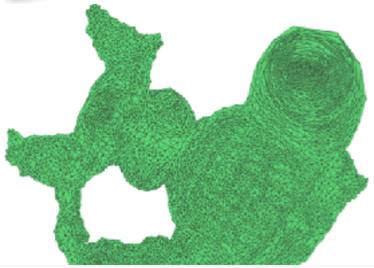

Snow samples micro-CT images
gridstep $h \approx 8\ \mu m$
Volume $\approx 50\ mm^3$
Mean curvatures H
Facetted snow
Fresh snow
Wet snow

Calculus on lattice surfaces

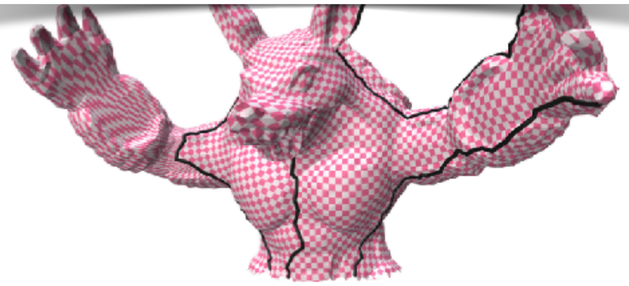


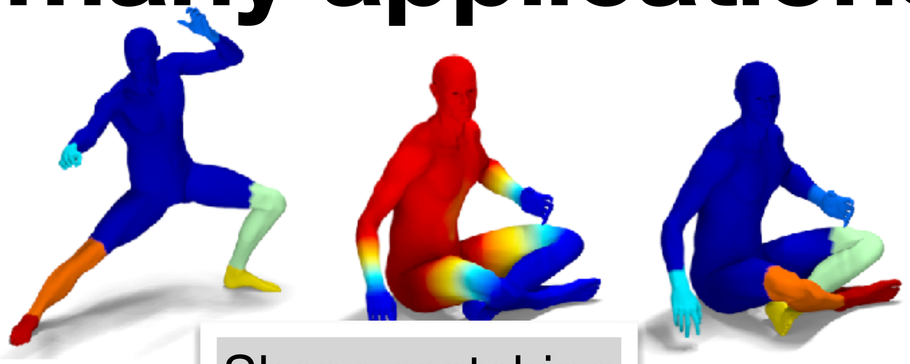
Calculus has many applications

Stripe patterns

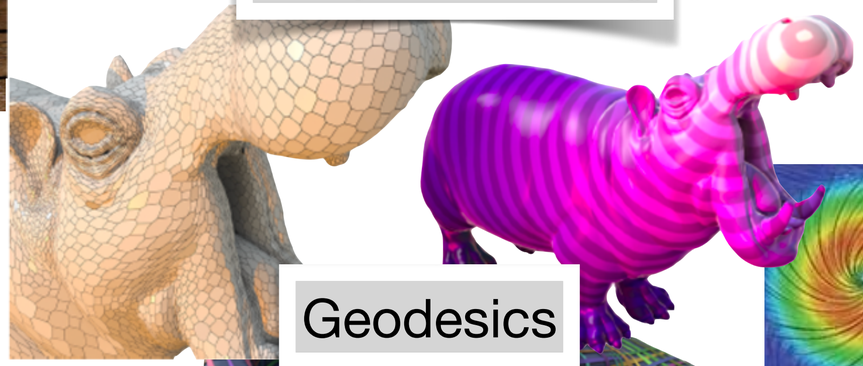


UV-parametrisation + cuts

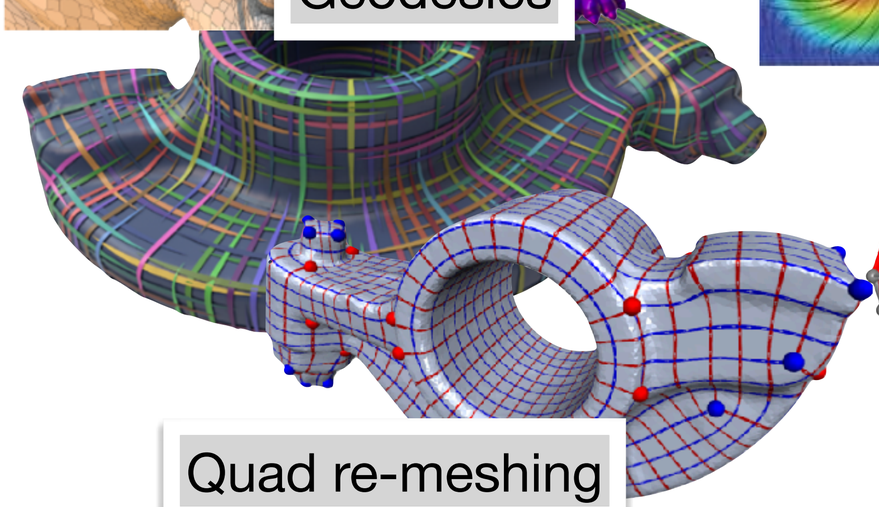




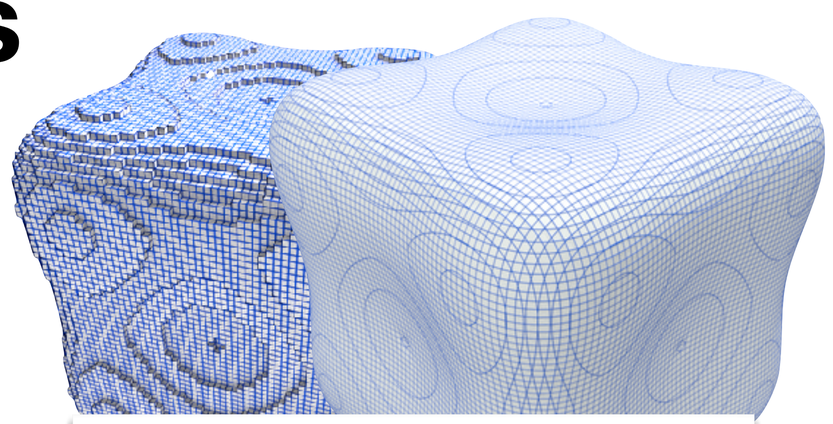
Shape matching



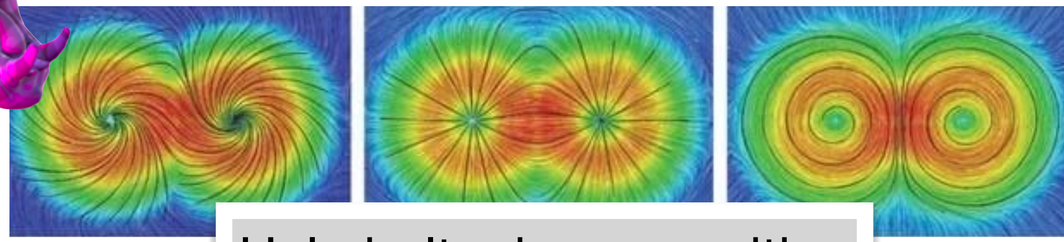
Geodesics



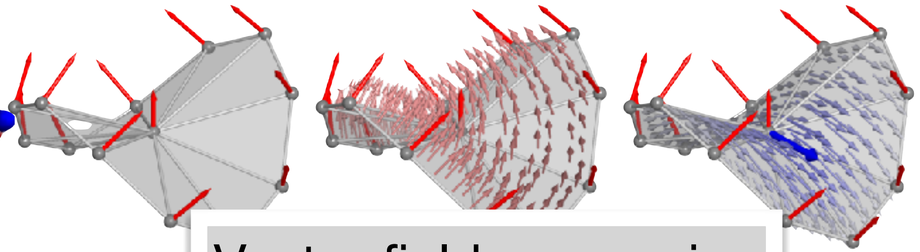
Quad re-meshing



Reconstruction minimising squared curvature



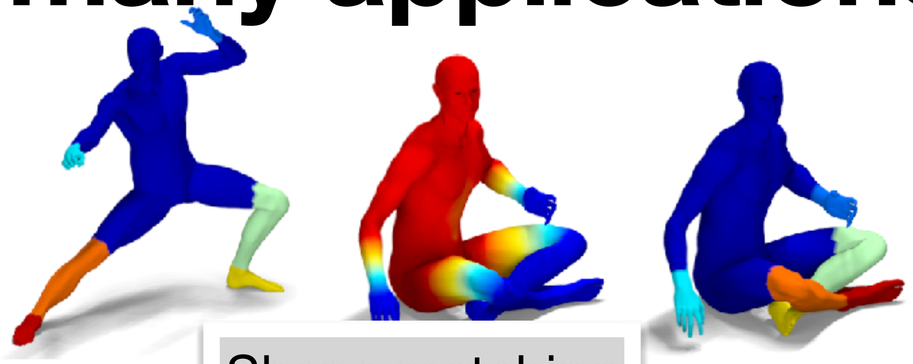
Helmholtz decomposition



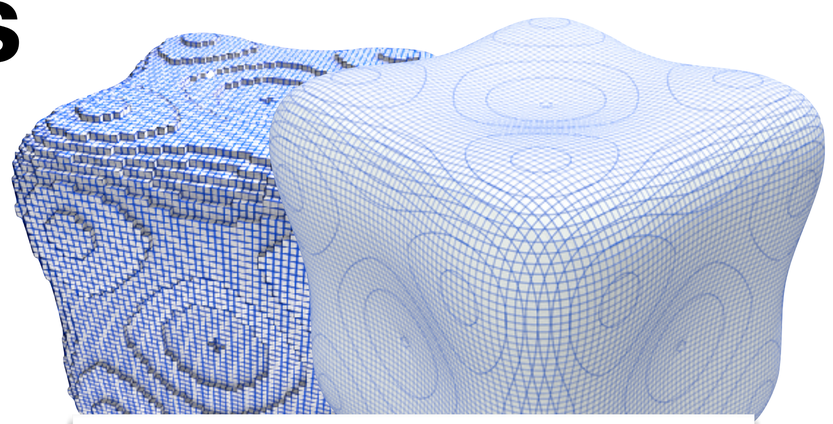
Vector field processing

Calculus has many applications

Stripe patterns



Shape matching



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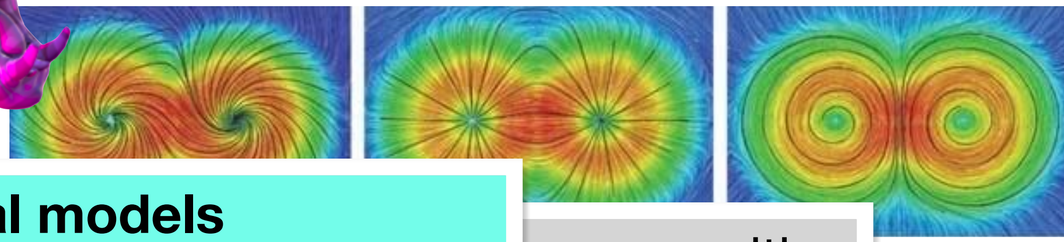
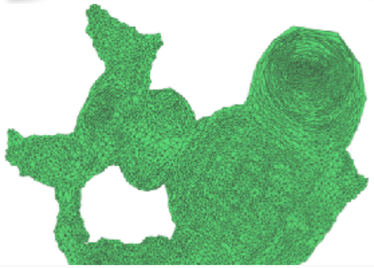
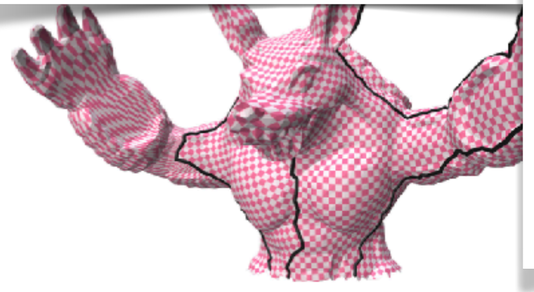
Geodesics



ecomposition



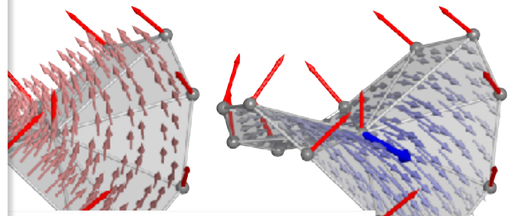
UV-parametrisation +



Solving PDE, minimising variational models involve differential operators on scalar or vector fields

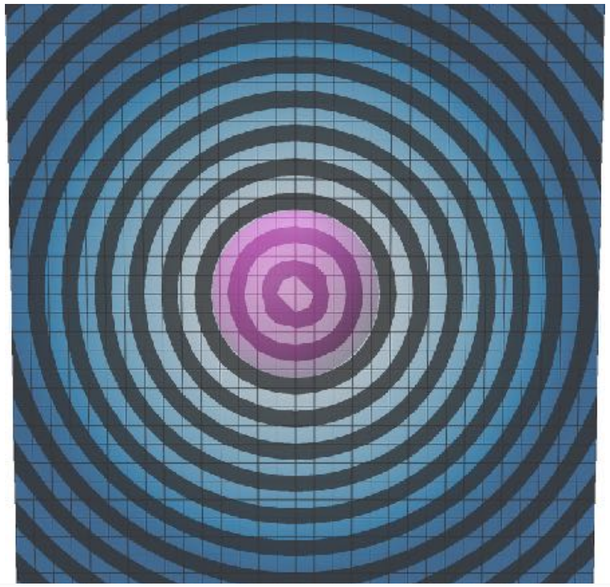
- gradient ∇, divergence $\nabla \cdot$, curl $\nabla \times$
- Laplace-Beltrami Δ, connections
- differential forms, exterior derivative,
- surface integrals

Well understood on triangulated or polygonal meshes



eld processing

Laplacian diffusion is not isotropic on digital surfaces



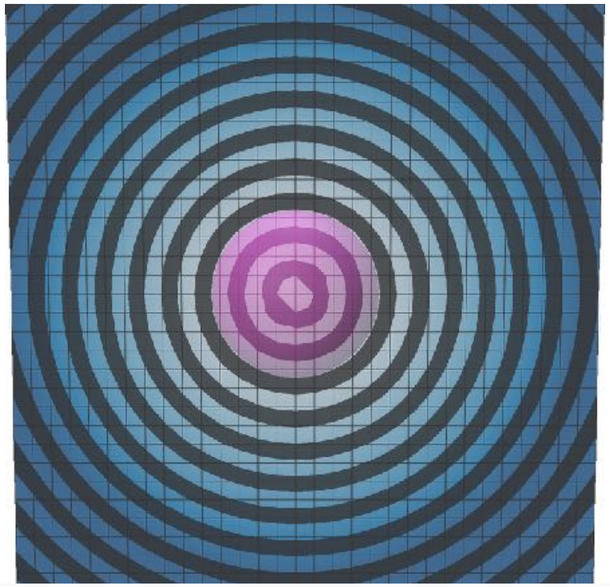
Heat method: $\mathbf{u}(0,\cdot) = \begin{cases} 1 \text{ on sources} \\ 0 \text{ elsewhere} \end{cases}$

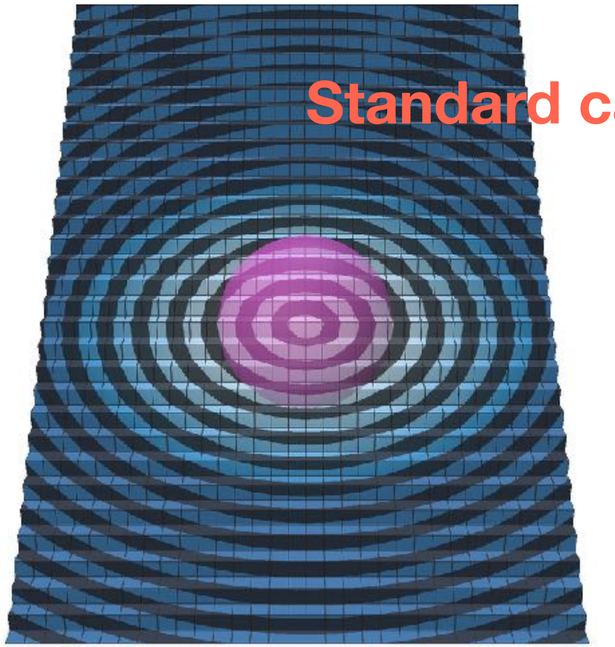
1) Integrate $\frac{\partial \mathbf{u}}{\partial t} = \Delta \mathbf{u}$ for some time δt

2) Evaluate vector field $\overrightarrow{\mathbf{X}} := -\frac{\nabla \mathbf{u}(\delta t)}{\|\nabla \mathbf{u}(\delta t)\|}$

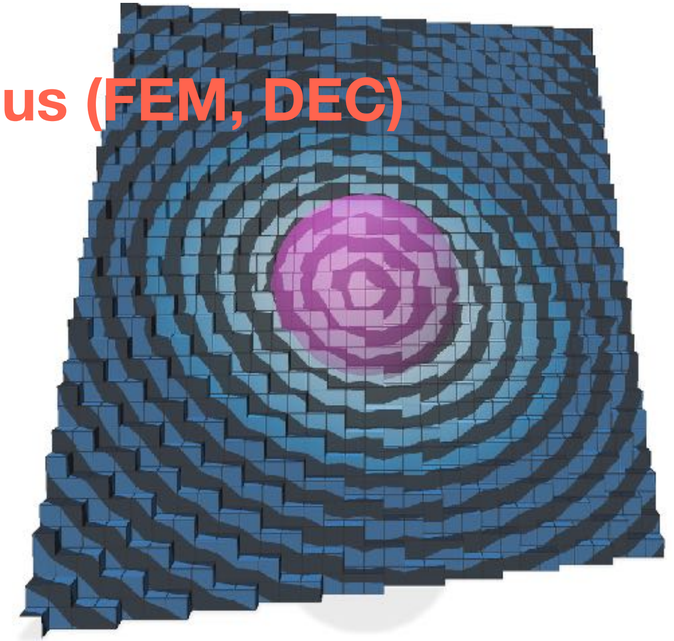
3) Distance $\mathbf{d}$: solve $\Delta \mathbf{d} = \nabla \cdot \overrightarrow{\mathbf{X}}$ (Poisson eq)

Laplacian diffusion is not isotropic on digital surfaces





Standard calculus (FEM, DEC)



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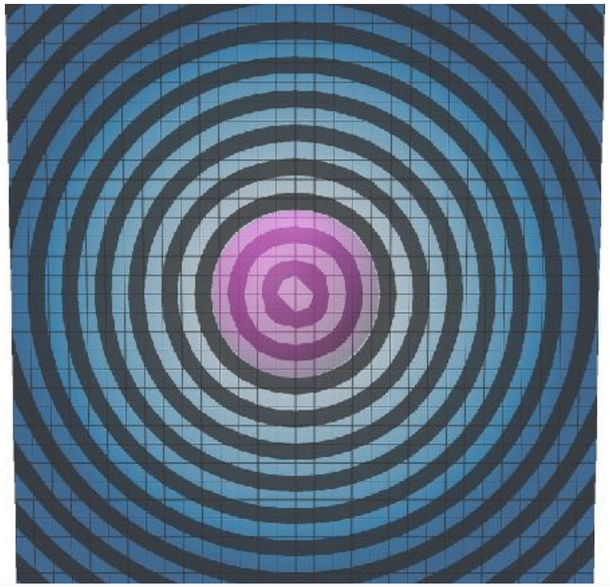

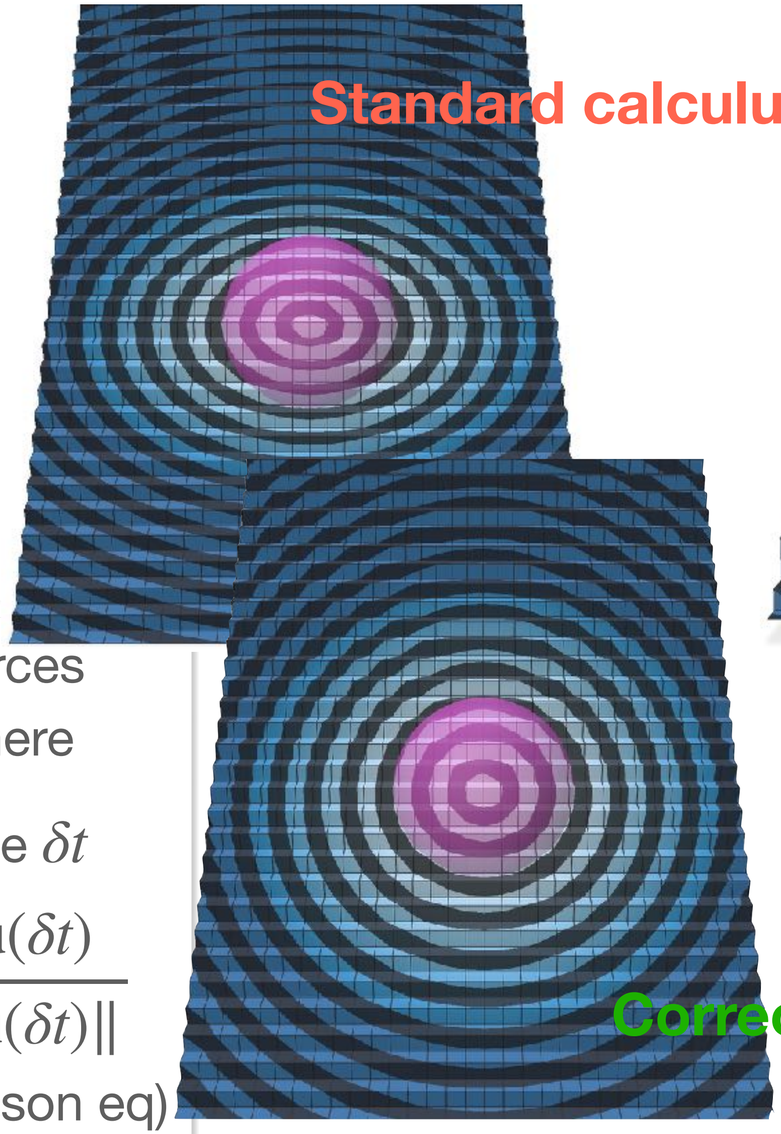

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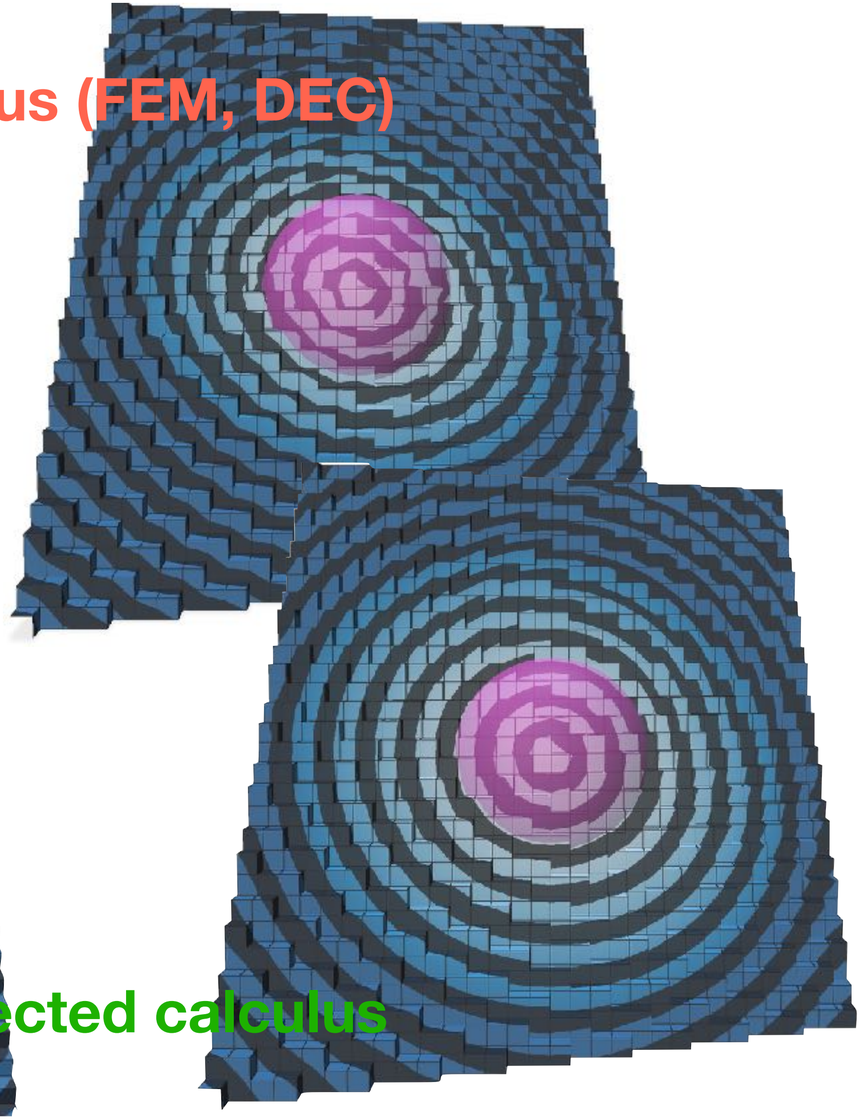

Corrected calculus

Heat method: $\mathbf{u}(0,\cdot) = \begin{cases} 1 \text{ on sources} \\ 0 \text{ elsewhere} \end{cases}$

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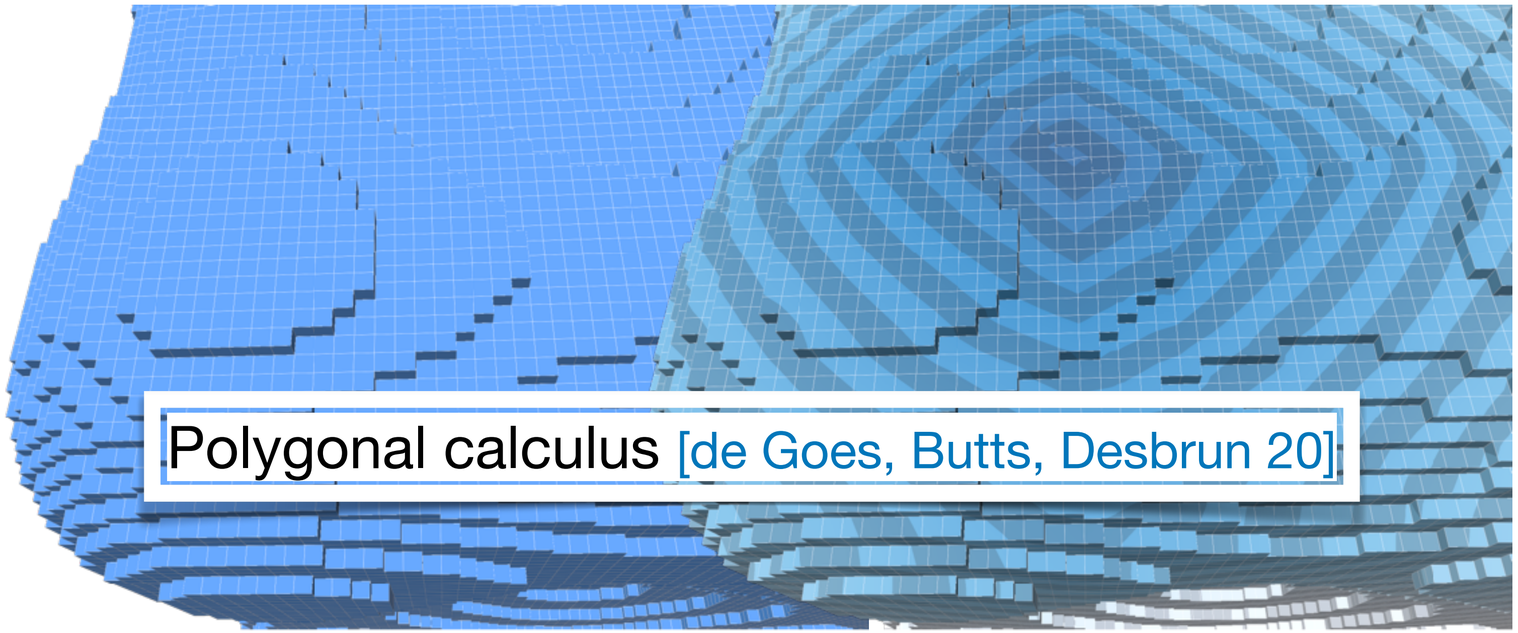
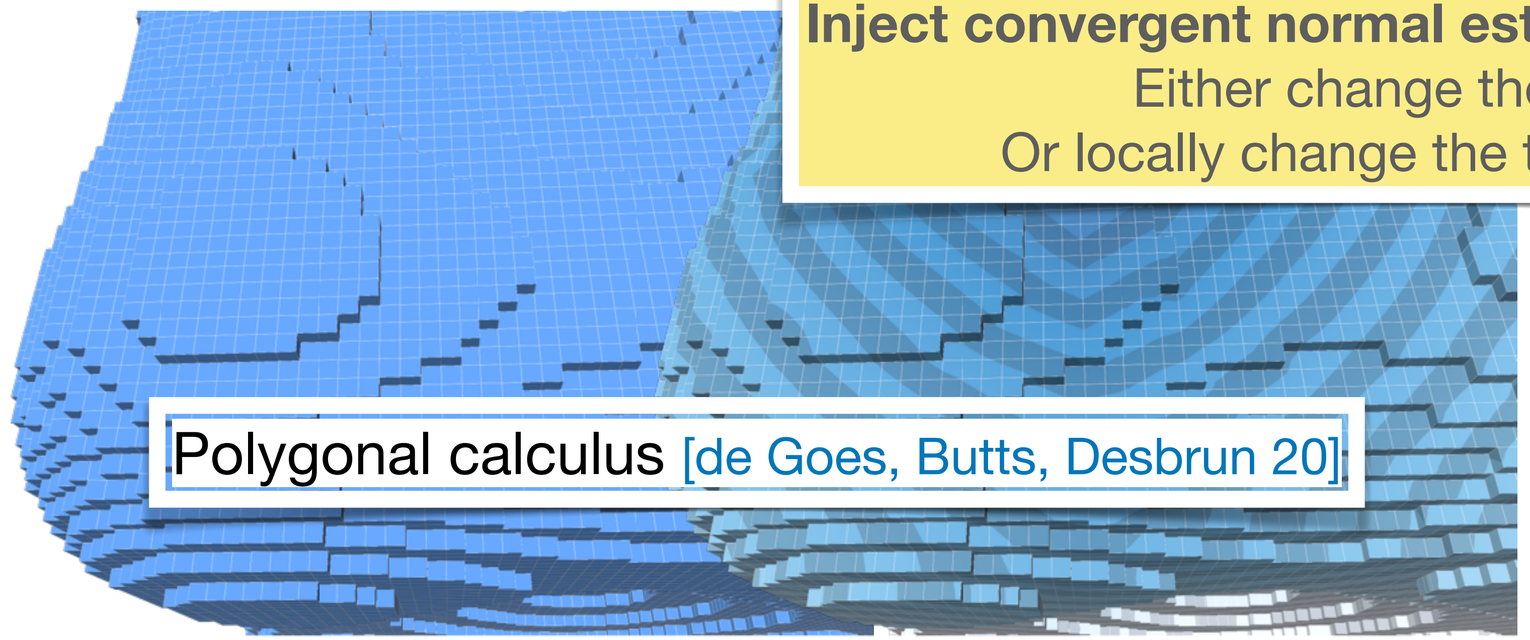
Adapting calculus on meshes to digital surfaces



Polygonal calculus [de Goes, Butts, Desbrun 20]

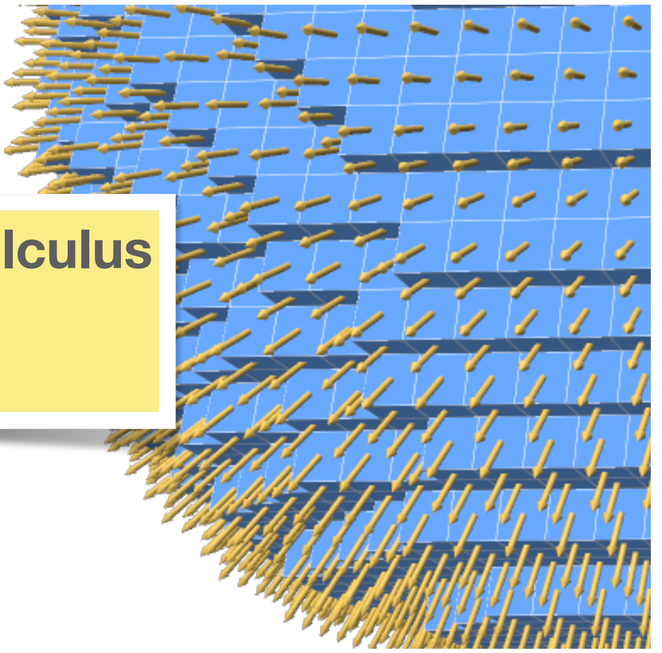
Adapting calculus on meshes to digital surfaces



Inject convergent normal estimations in calculus

Either change the metric

Or locally change the tangent plane



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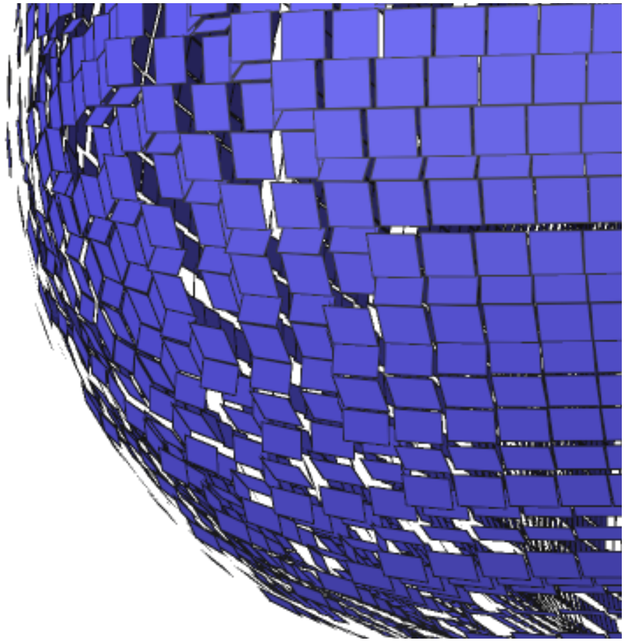
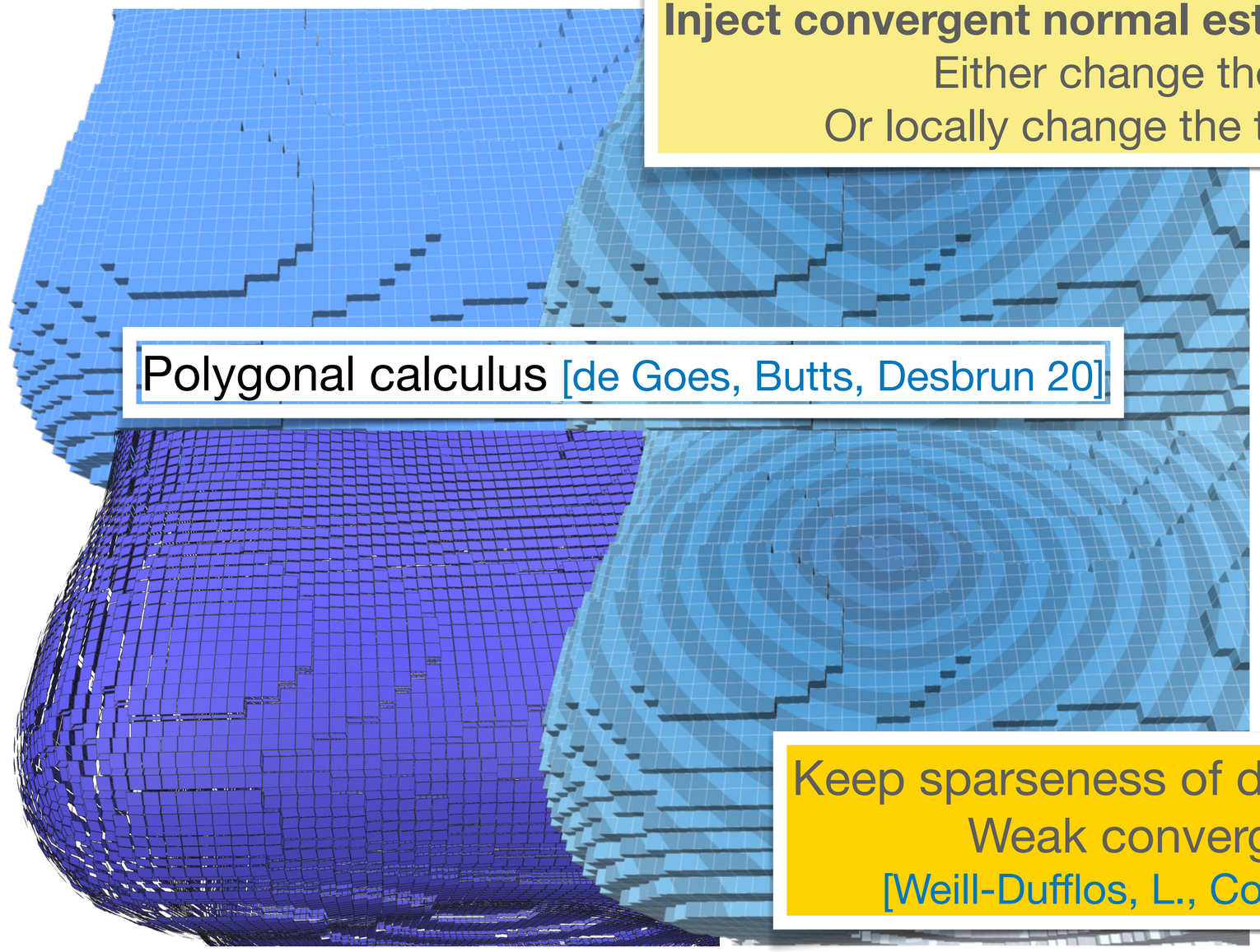
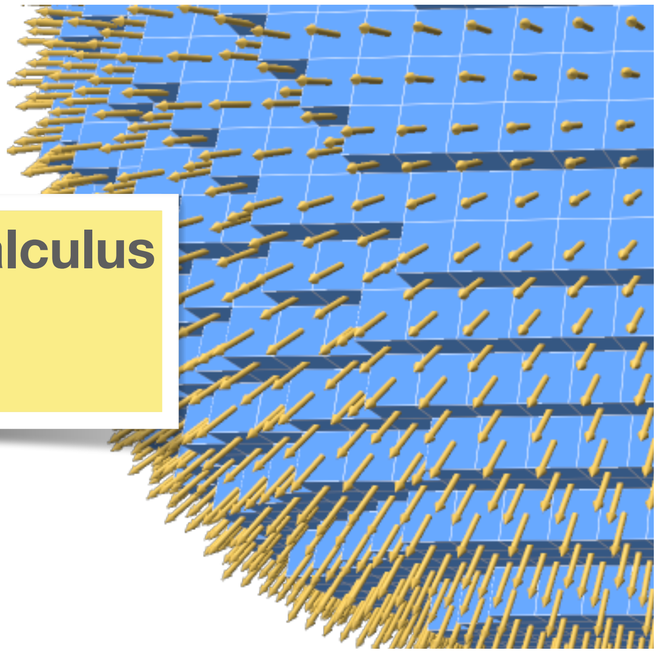


Adapting calculus on meshes to digital surfaces





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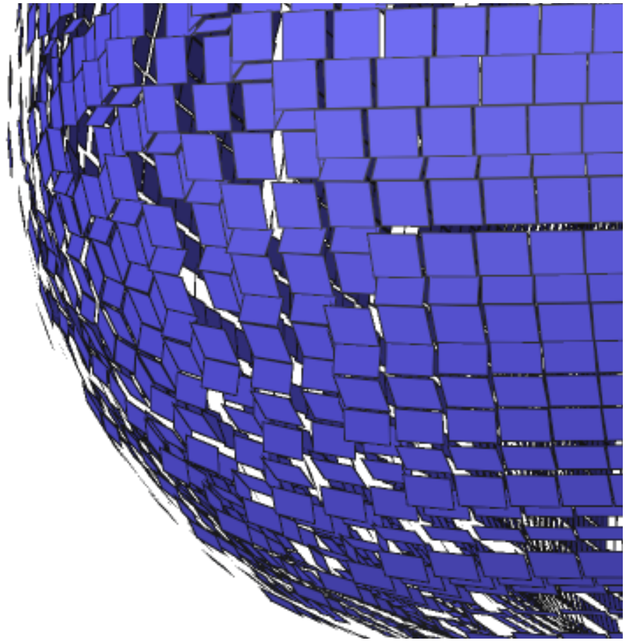
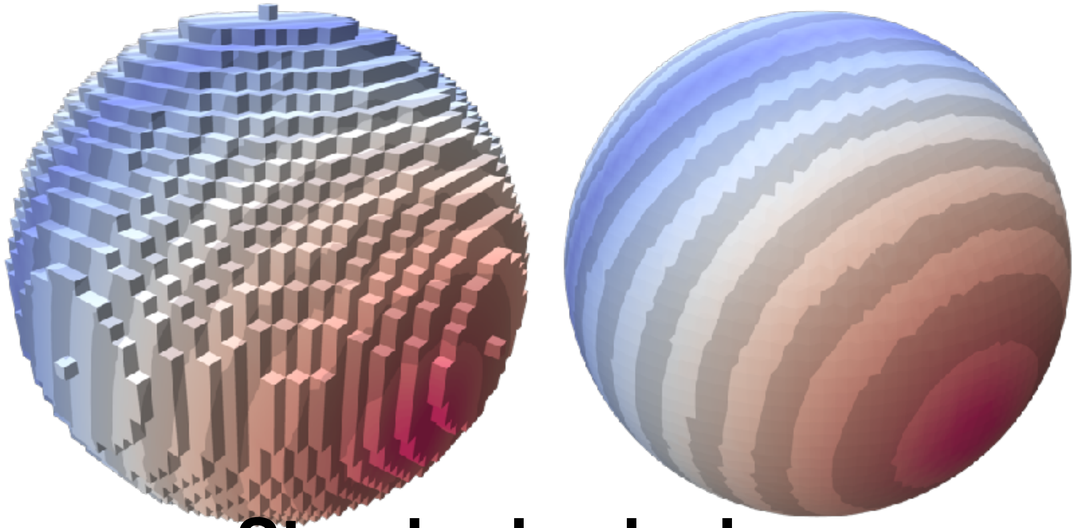


Keep sparseness of differential ops

Weak convergence

[Weill-Dufflos, L., Coeurjolly 24]

Spectral analysis: eigenvectors of Laplace-Beltrami Δ



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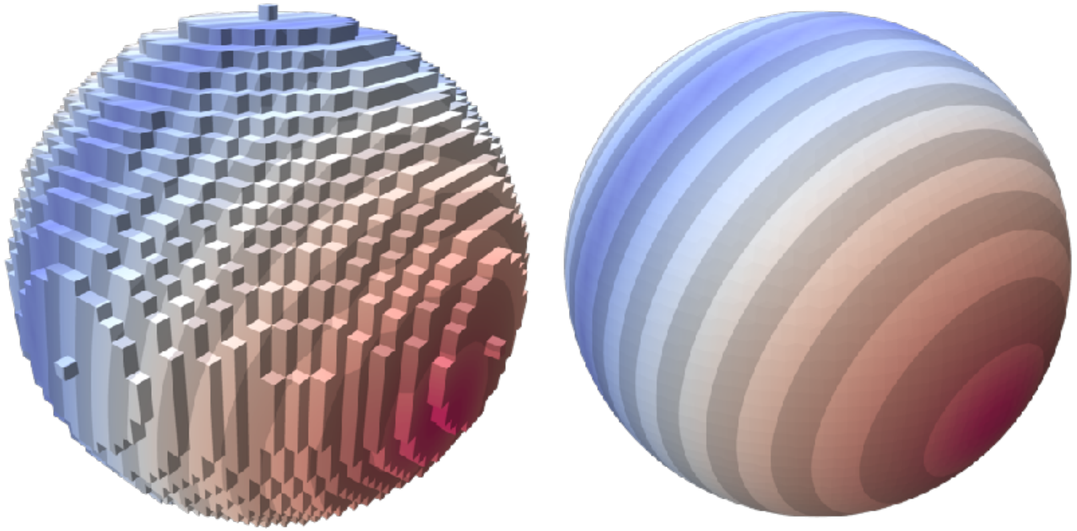
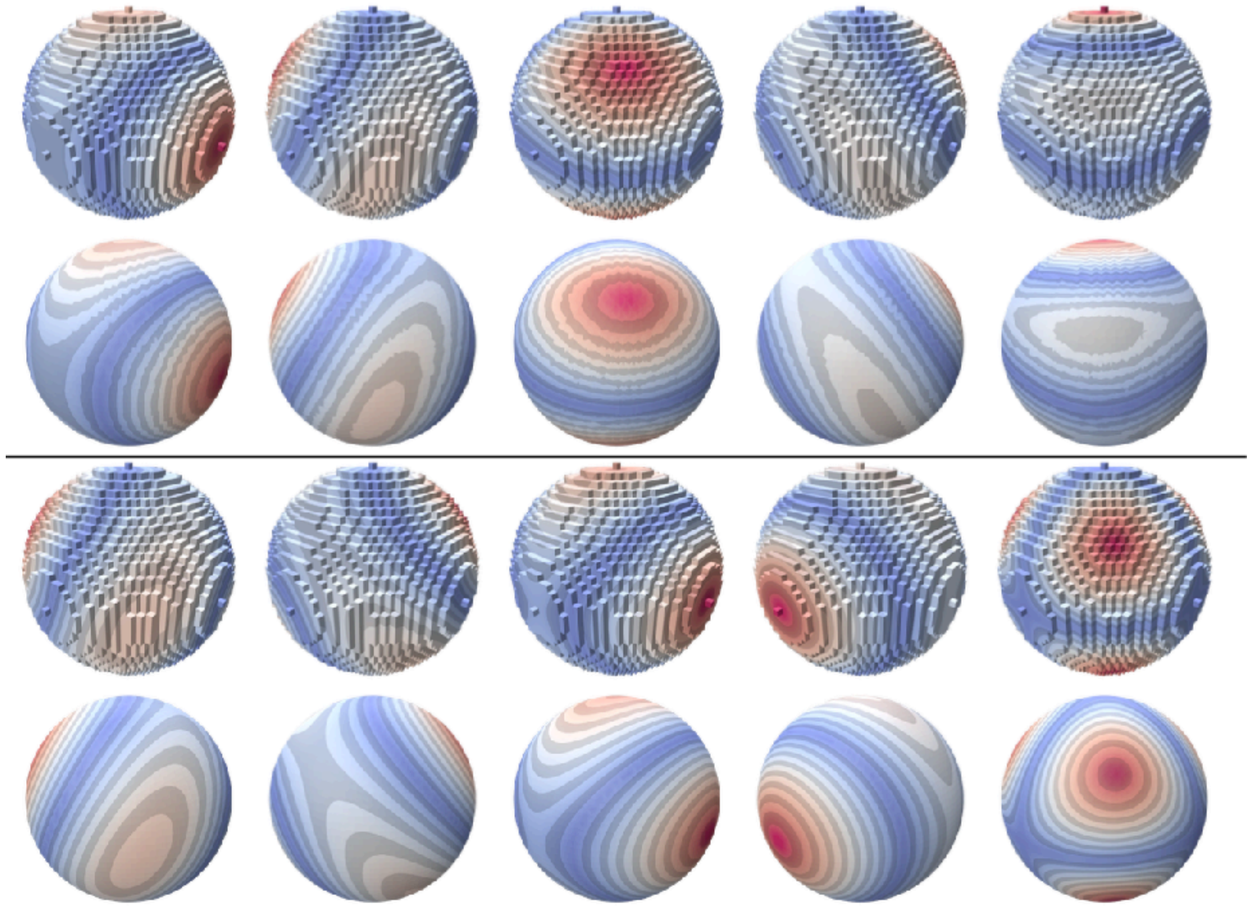


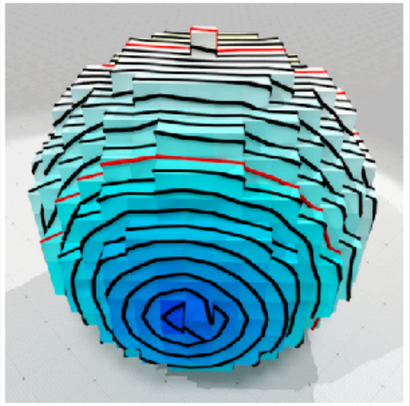
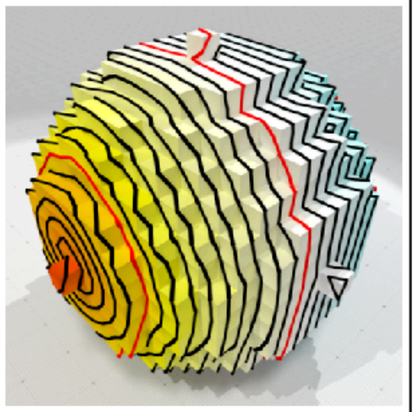
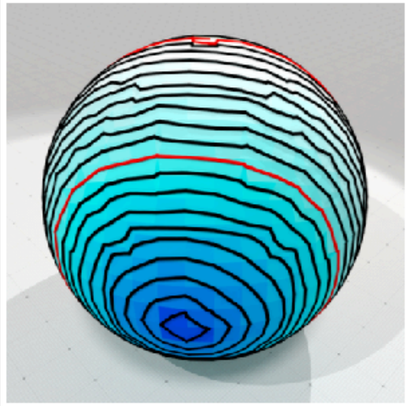
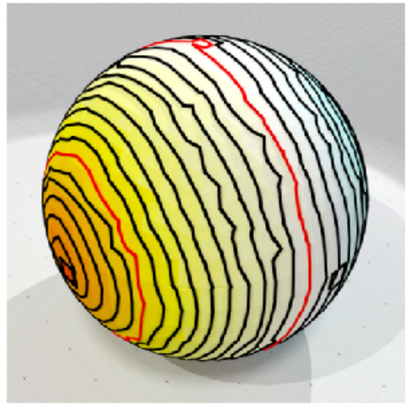
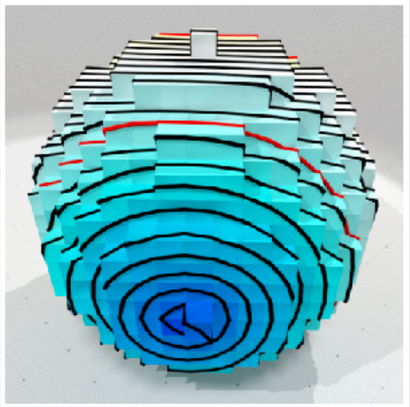
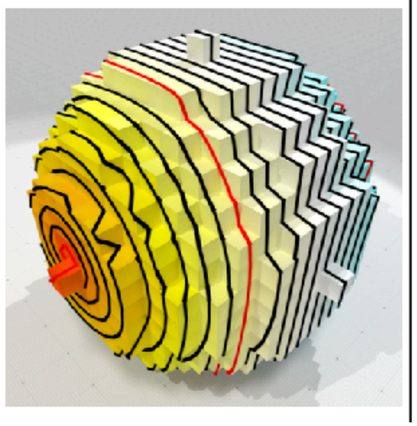
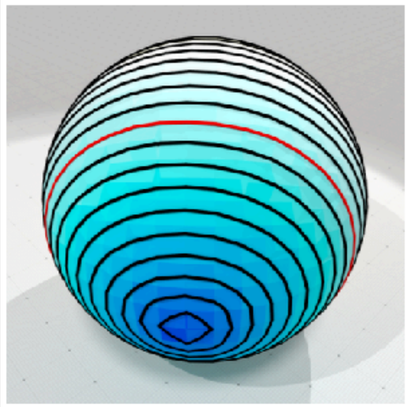
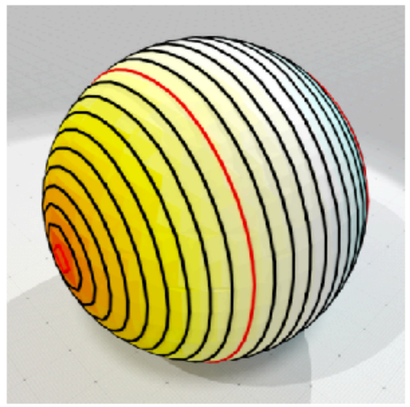
Corrected calculus

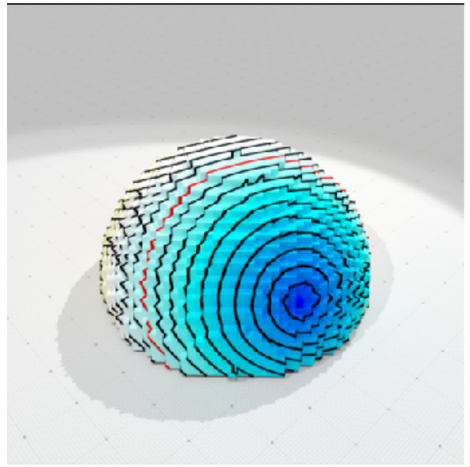


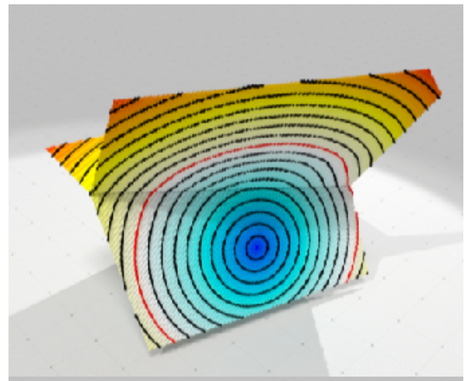
Some solutions to $\Delta \mathbf{u} = \lambda \mathbf{u}$

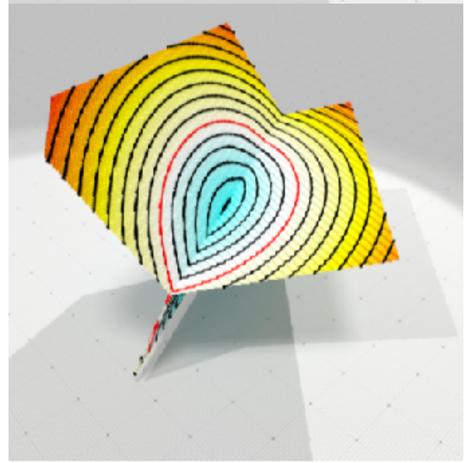
[Coeurjolly, L. 22]

Geodesics using the heat method

	Sphere of radius 1, digitized at gridstep $h = 0.125$			
	distance on digitized surface		distance projected on smooth surface	
	front view	back view	front view	back view
Standard $d_{\max} = 3.704$				
Corrected $d_{\max} = 3.068$				

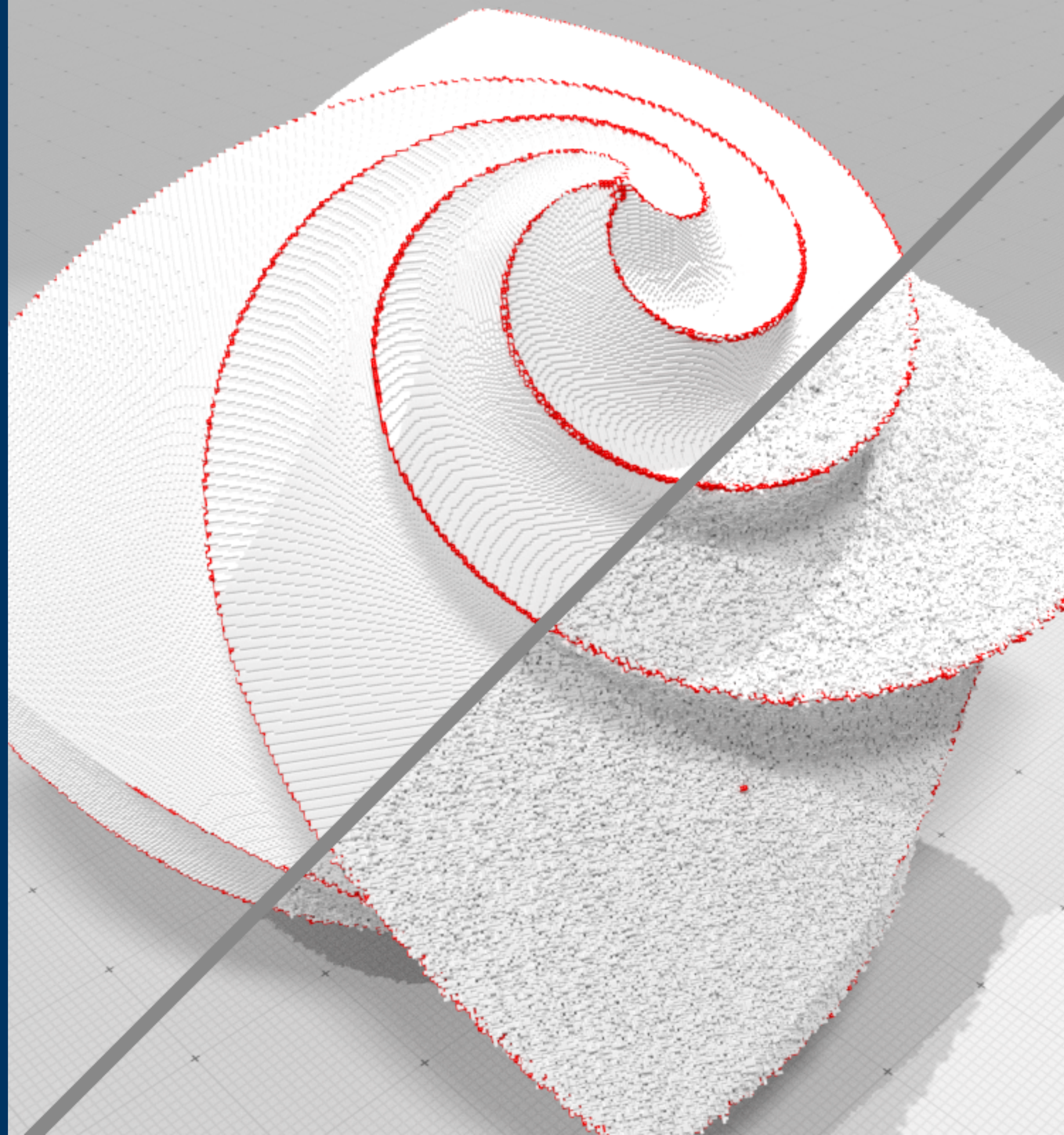






[Coeurjolly, L. 22]

Examples of geometry processing with Calculus



Piecewise smooth reconstruction

Step 1: g = normal estimation (e.g. integral invariants)

Step 2: minimise $AT_\epsilon(u, v)$ (Ambrosio-Tortorelli functional) :

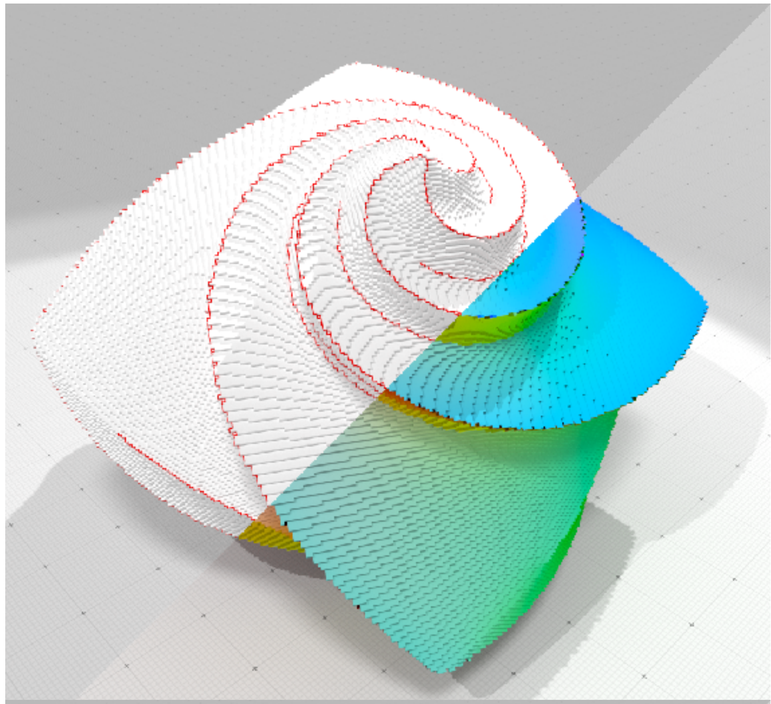
$$AT_\epsilon(u,v) = \alpha \int_M |u - g|^2 dx + \int_M |v\nabla u|^2 + \lambda\epsilon |\nabla v|^2 + \frac{1}{4\epsilon}|1 - v|^2 dx$$

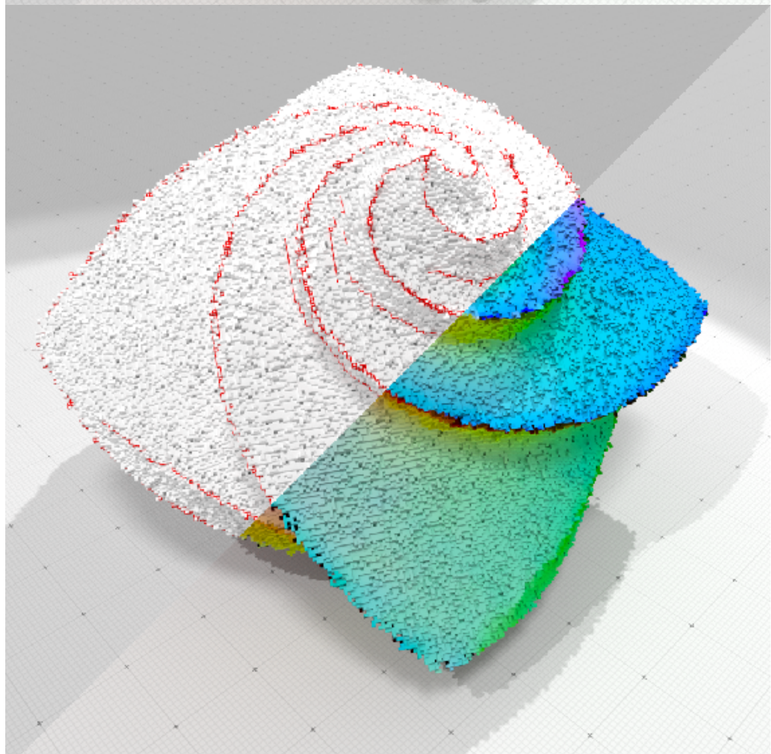
Functional is convex for fixed u or v

$\Rightarrow$ alternate minimization

u: piecewise smooth vector field

v: 0 on singularities, 1 elsewhere





[Coeurjolly, Gueth, L. 16]

Piecewise smooth reconstruction

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Reconstructed normals are close to the input ones

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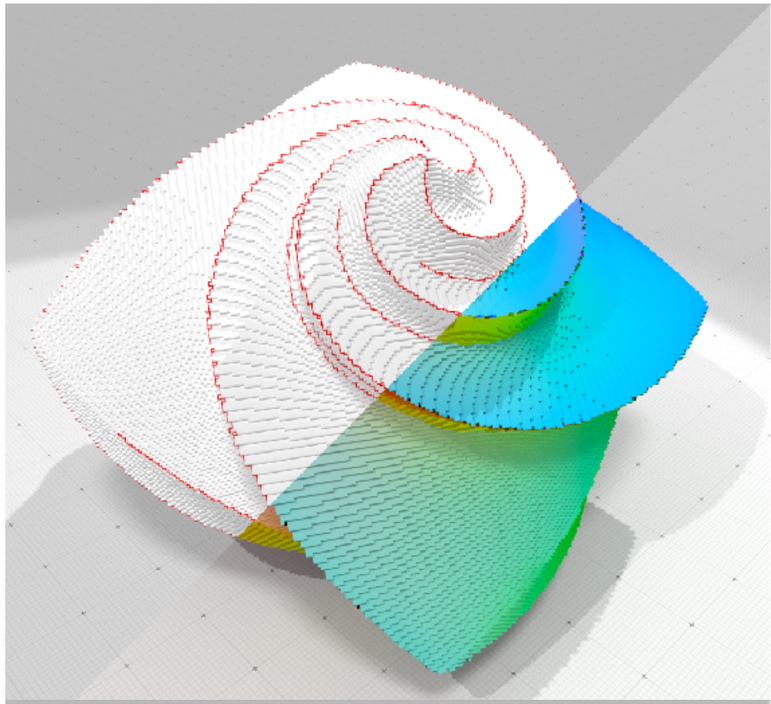

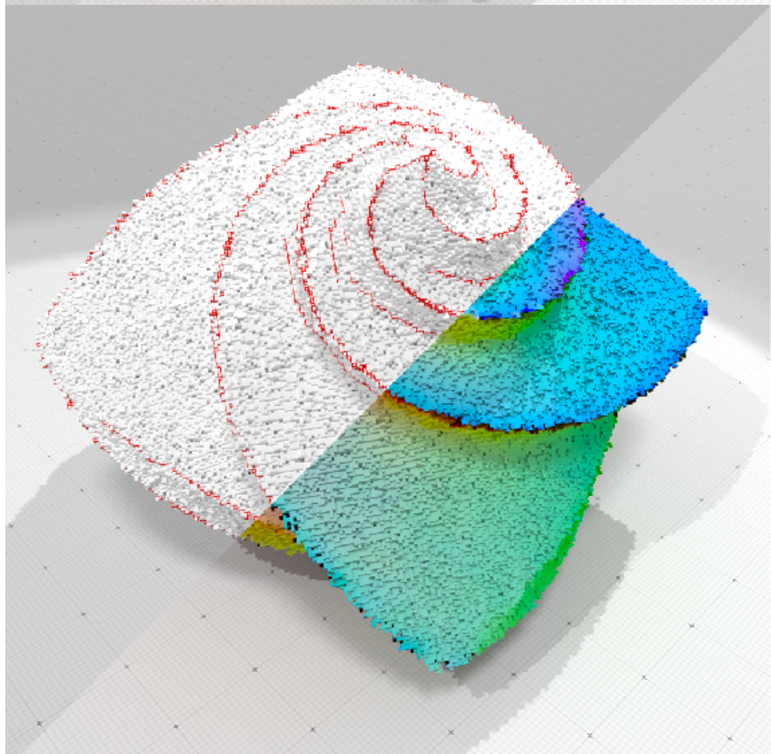

[Coeurjolly, Gueth, L. 16]

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Normal field must be smooth except at singularities v

Functional is convex for fixed u or v

$\Rightarrow$ alternate minimization

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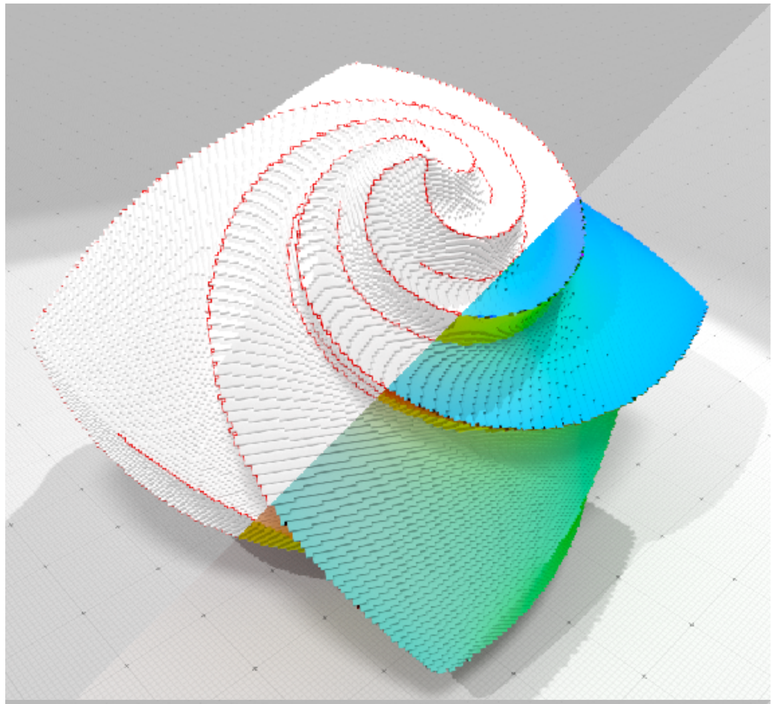

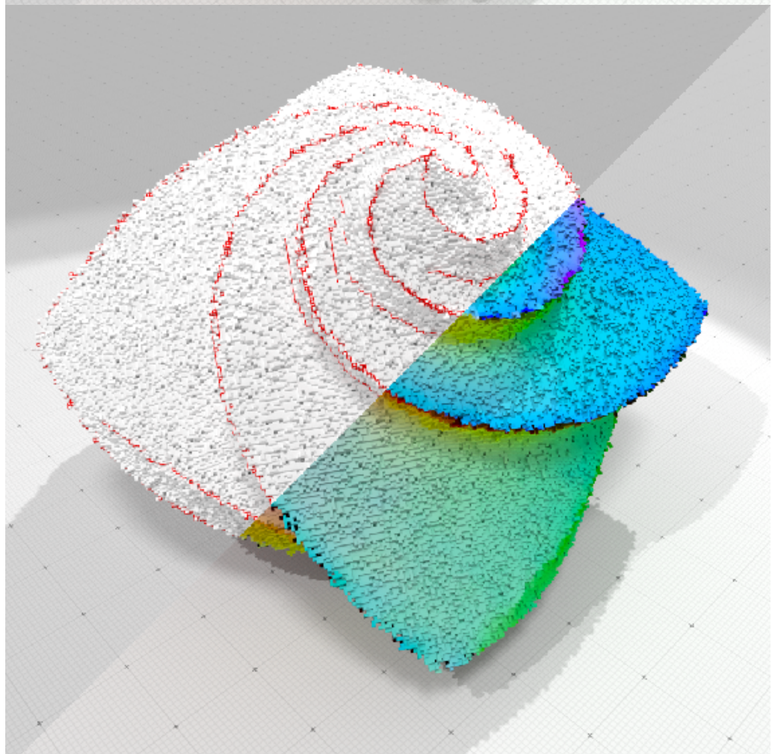

[Coeurjolly, Gueth, L. 16]

Piecewise smooth reconstruction

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Reconstructed normals are close to the input ones

Normal field must be smooth except at singularities v

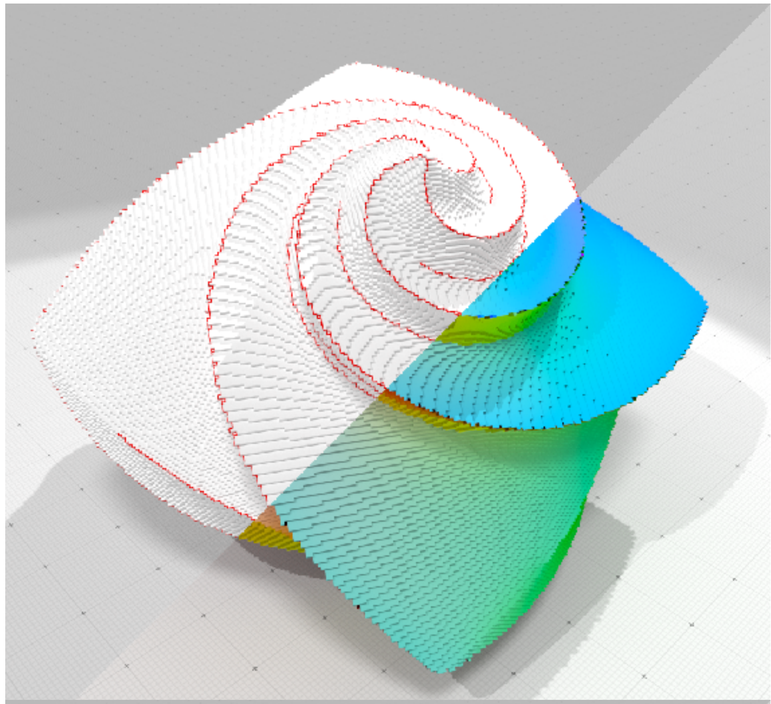
Penalizes the *length* of singularities

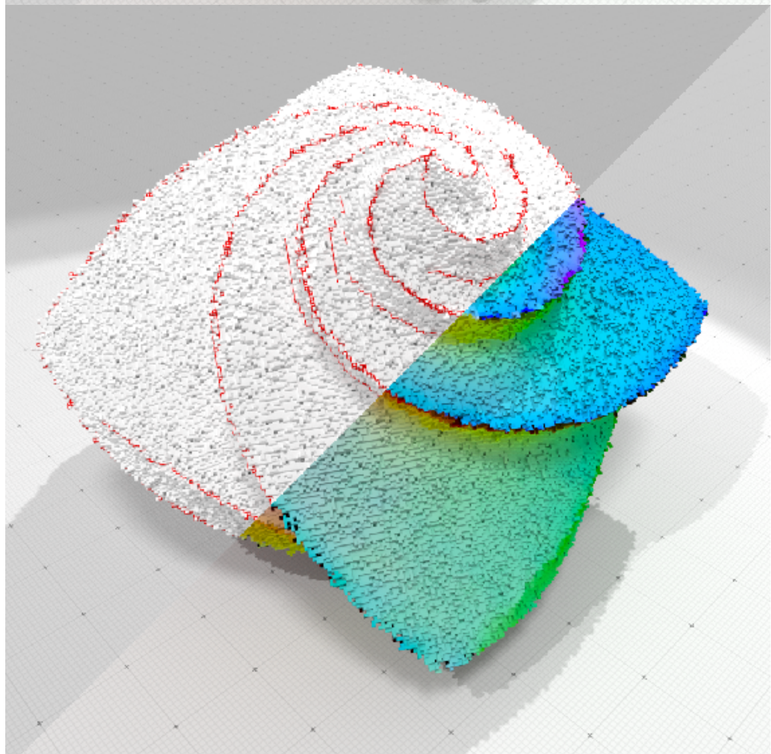
Functional is convex for fixed u or v

$\Rightarrow$ alternate minimization

u: piecewise smooth vector field

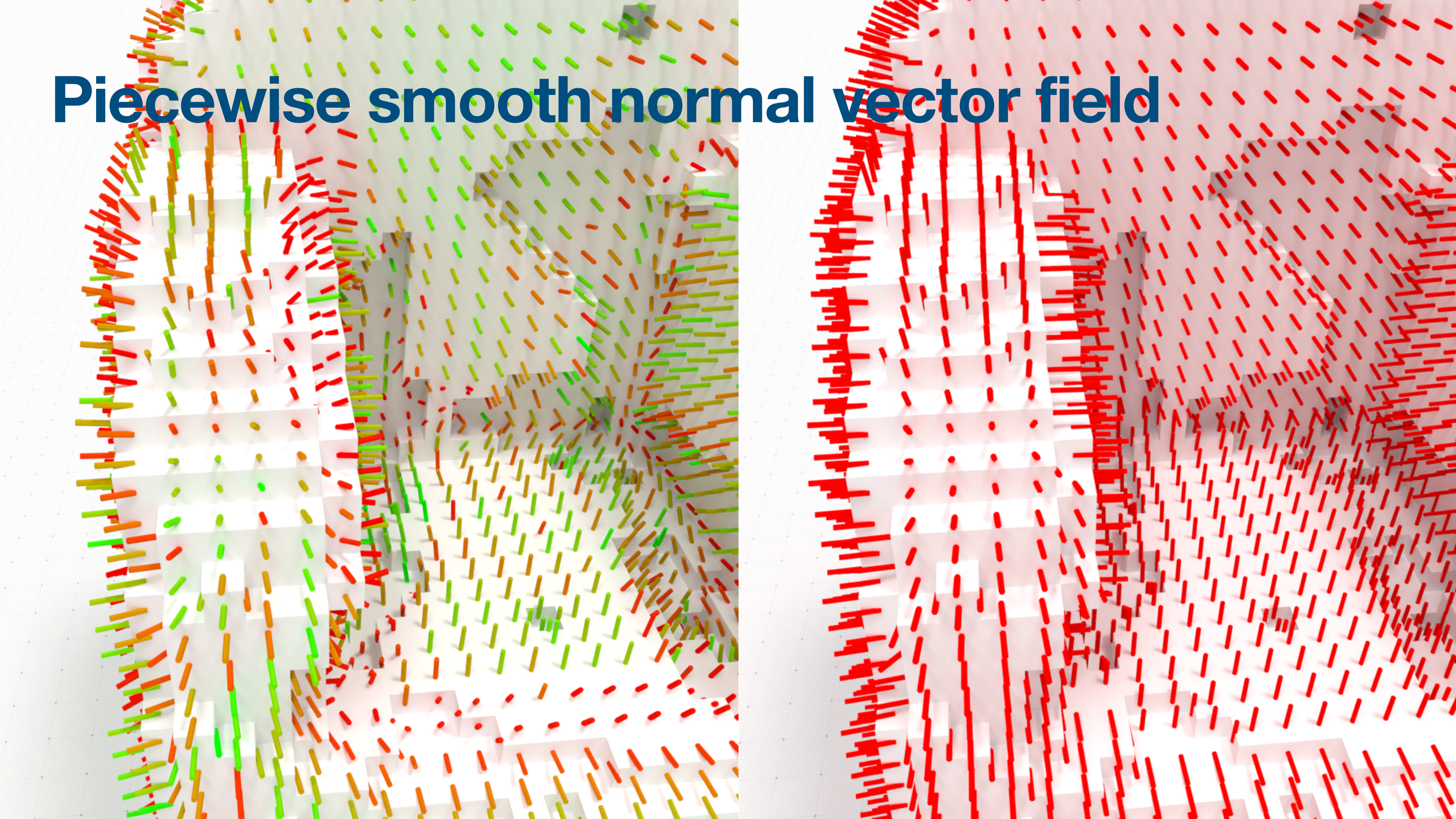
v: 0 on singularities, 1 elsewhere





[Coeurjolly, Gueth, L. 16]

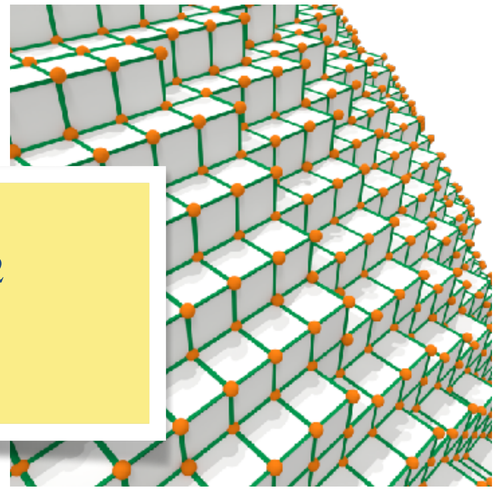
Piecewise smooth normal vector field

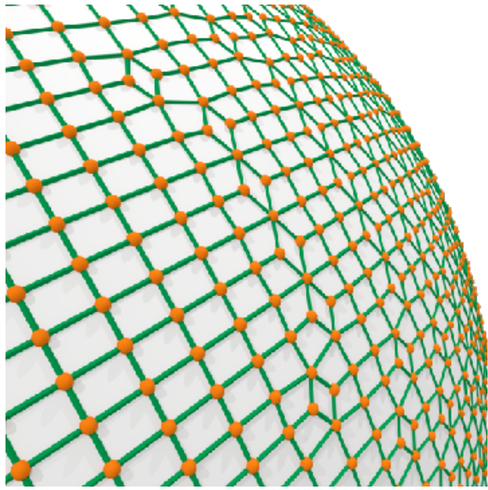


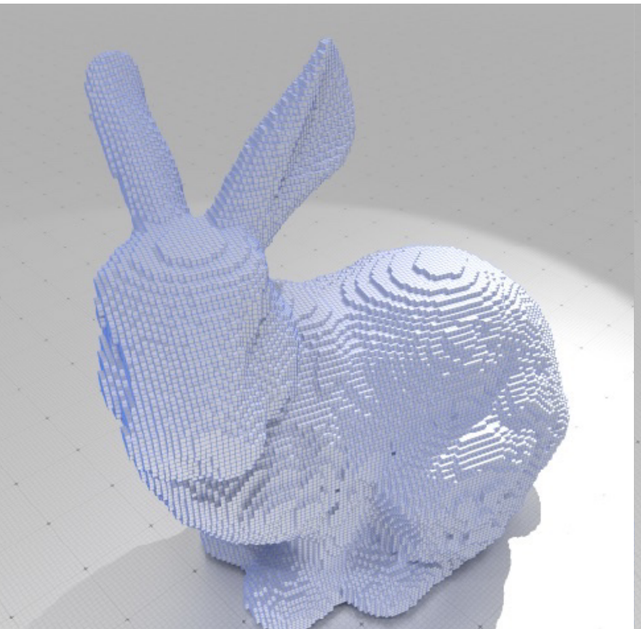
Piecewise smooth reconstruction

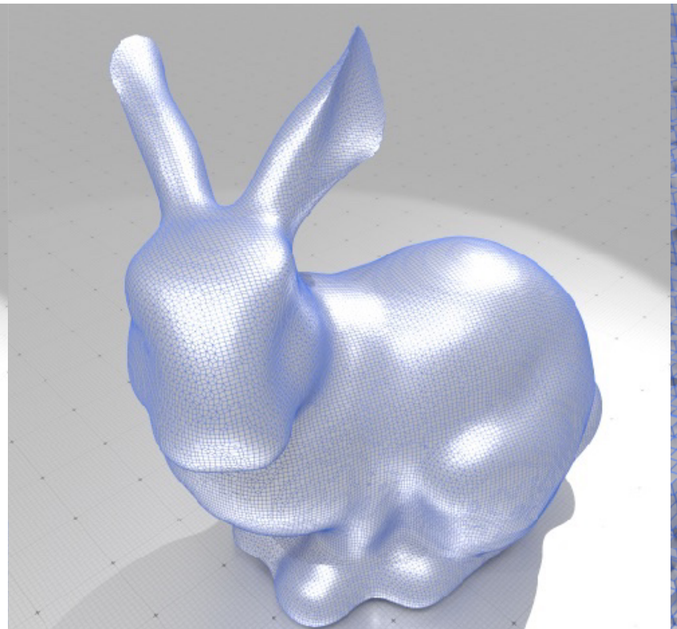
Step 3: Minimise cost $\mathscr{E}(\hat{P})$ by gradient descent

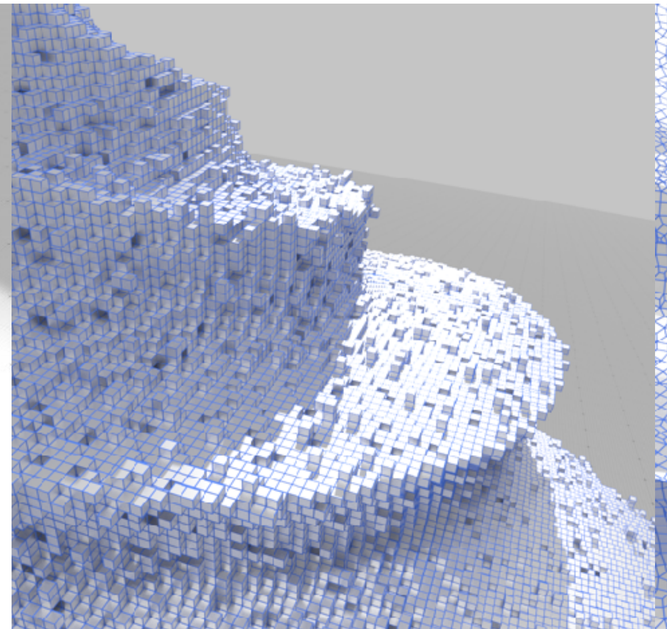
$$\mathscr{E}(\hat{P}) := \alpha \sum_{i=1}^{n} \|\mathbf{p}_i - \hat{\mathbf{p}}_i\|^2 + \beta \sum_{f \in F} \sum_{\hat{\mathbf{e}}_j \in \partial f} (\hat{\mathbf{e}}_j \cdot \mathbf{n}_f)^2 + \gamma \sum_{i=1}^{n} \|\hat{\mathbf{p}}_i - \hat{\mathbf{b}}_i\|^2$$

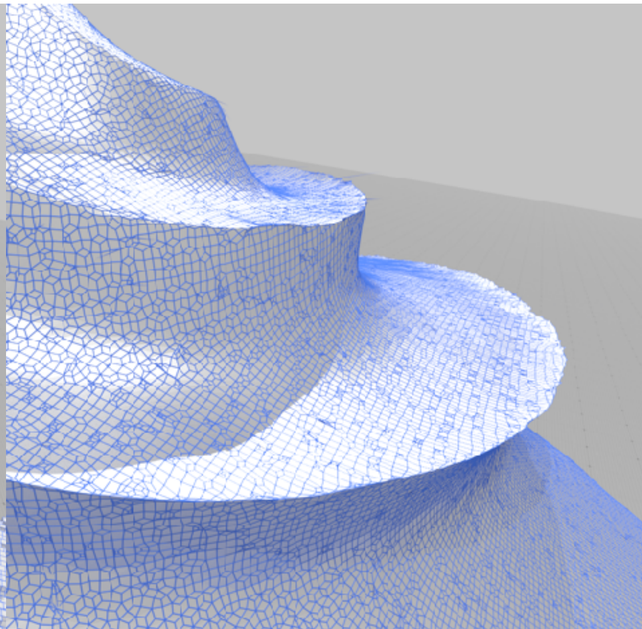












Piecewise smooth reconstruction

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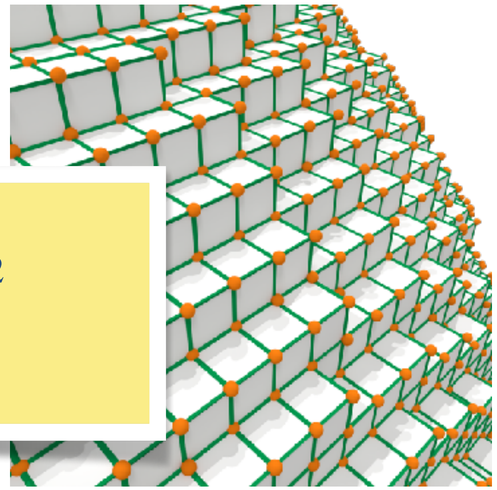
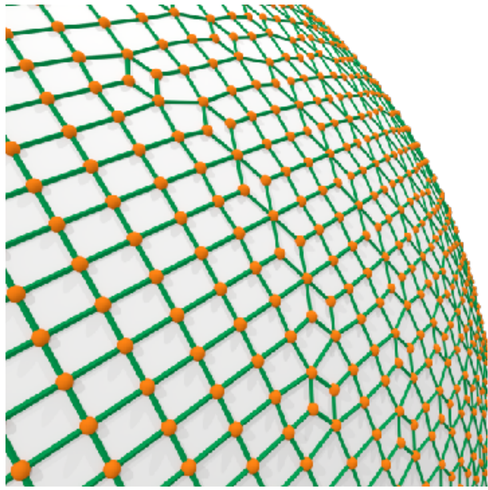
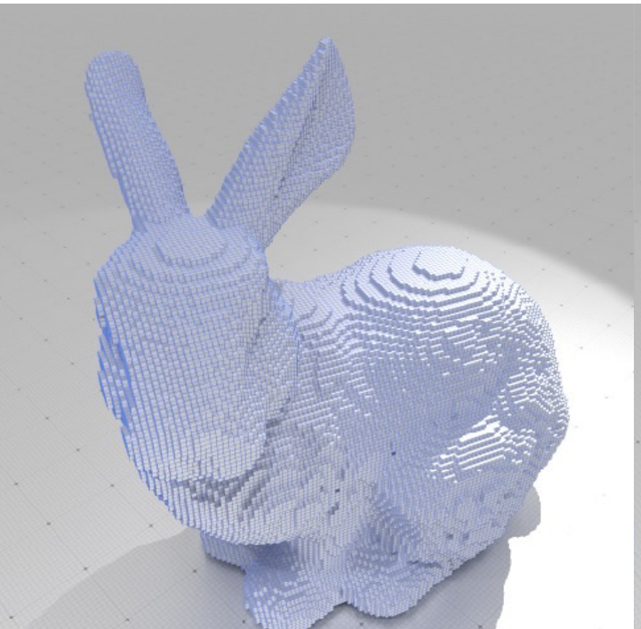
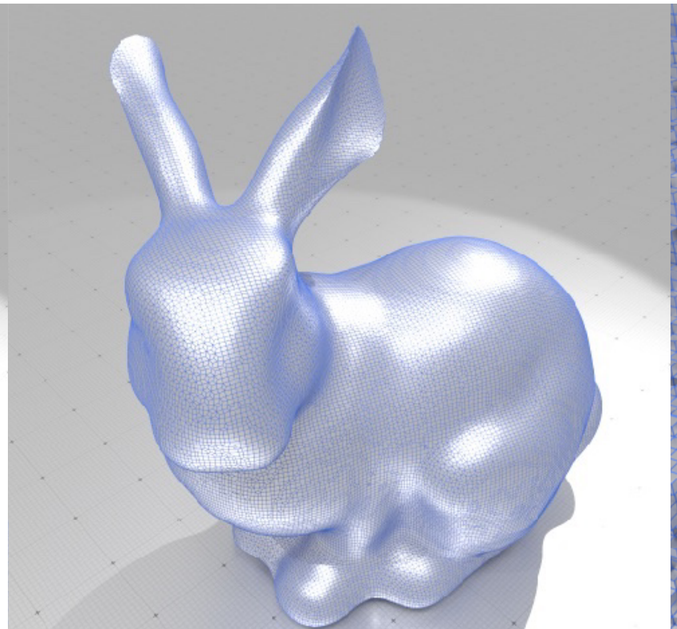
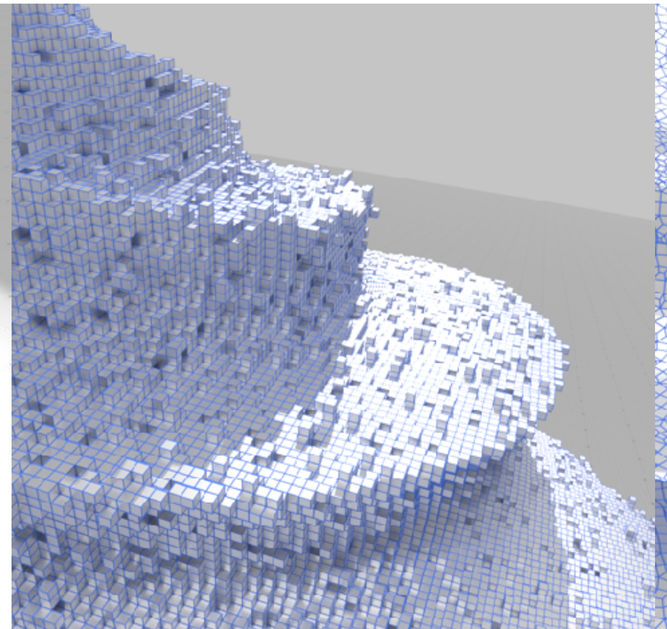
optimized vertices are not too far from original ones

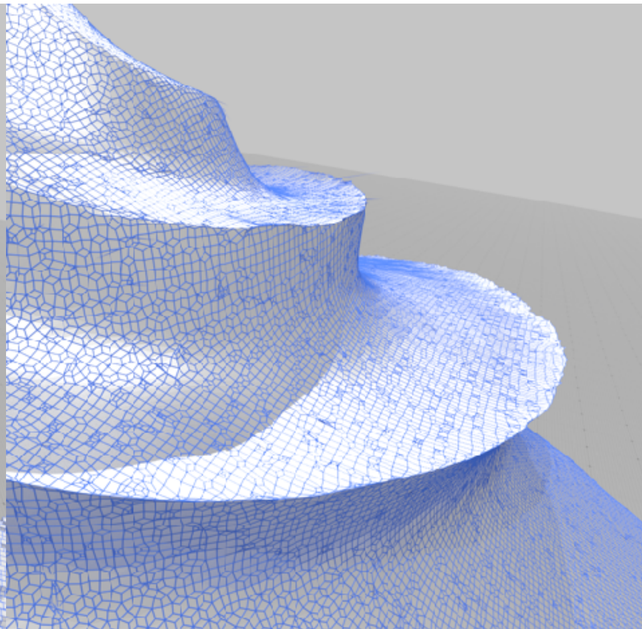












Piecewise smooth reconstruction

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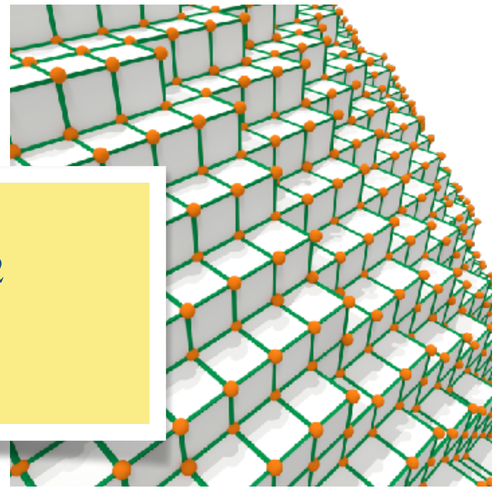
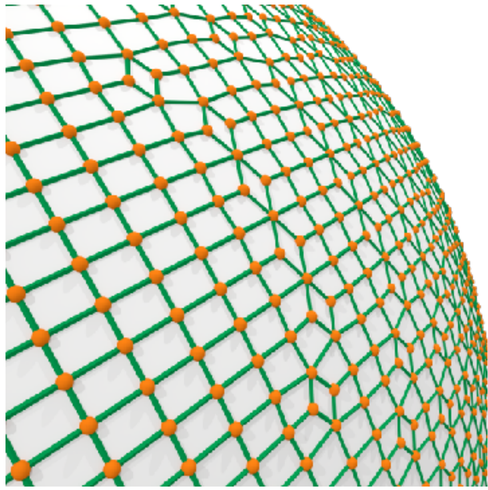
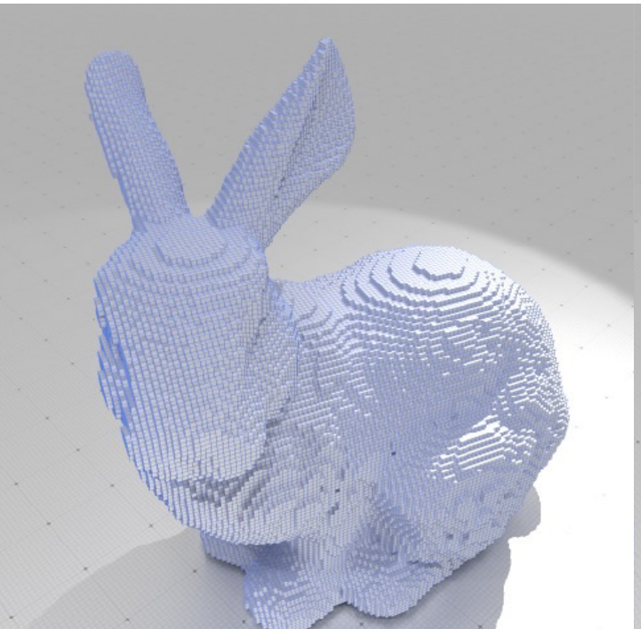
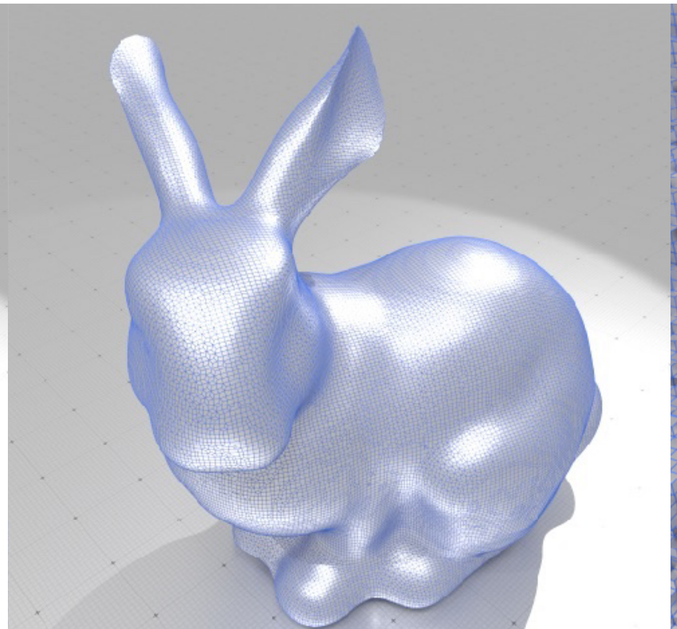
optimized vertices are not too far from original ones

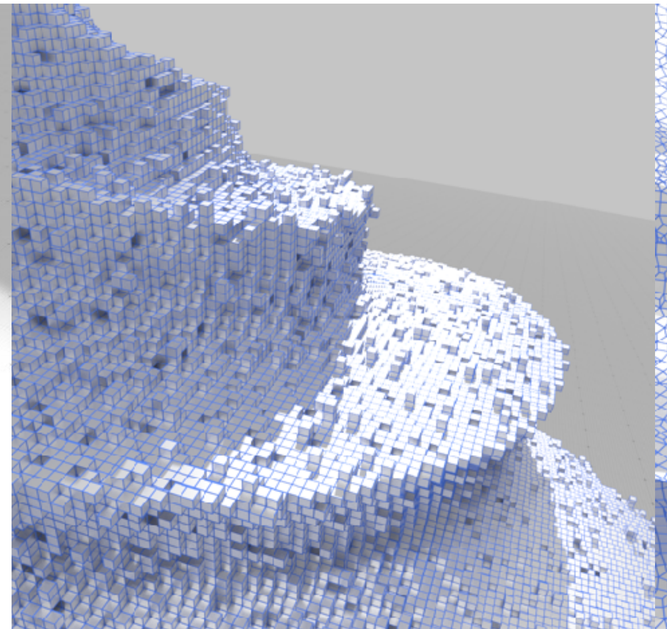
Edges must be as orthogonal as possible to the given normal vectors

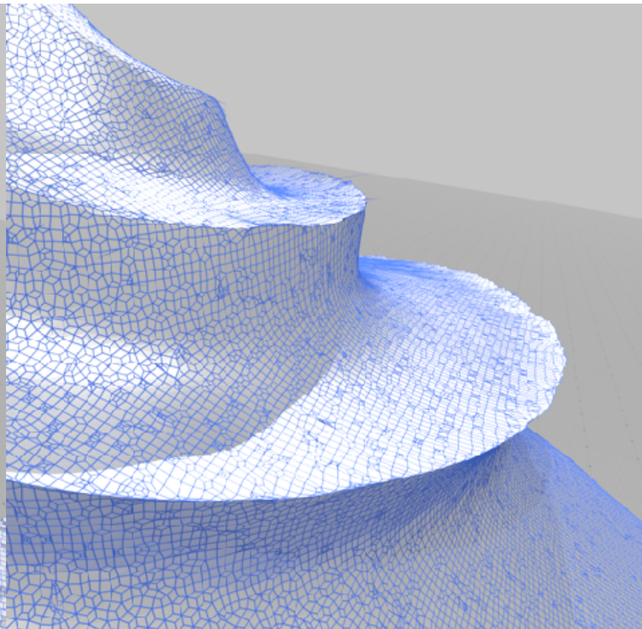












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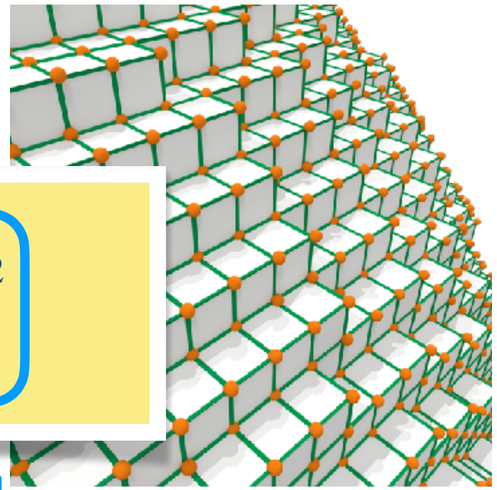
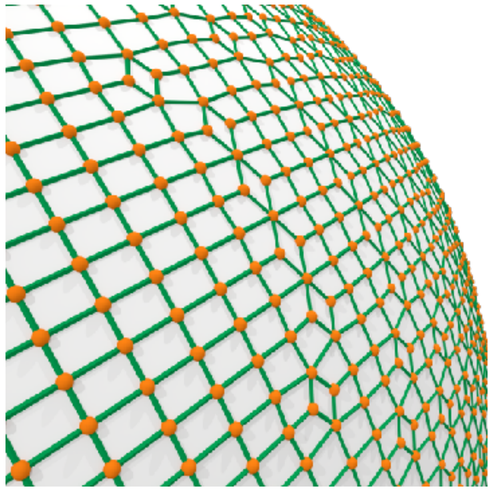
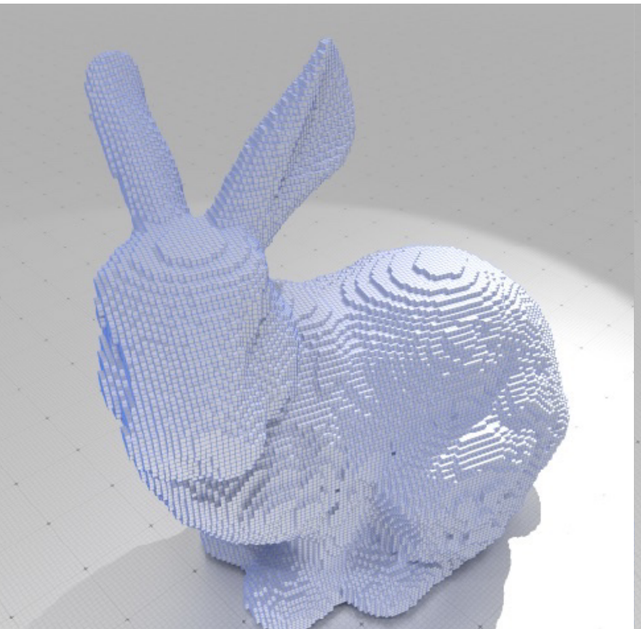
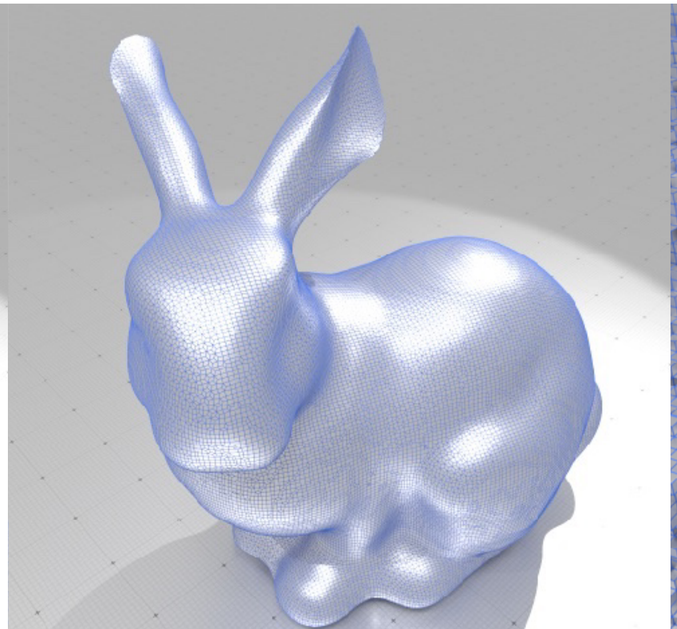
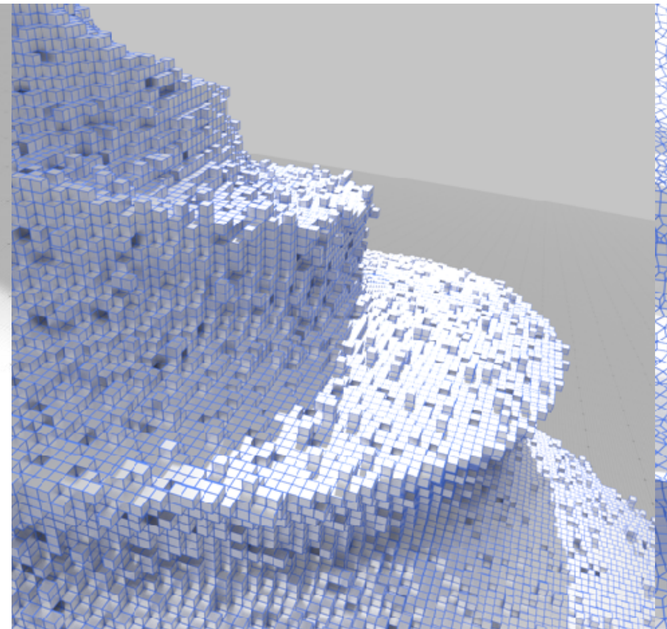
Fairness term

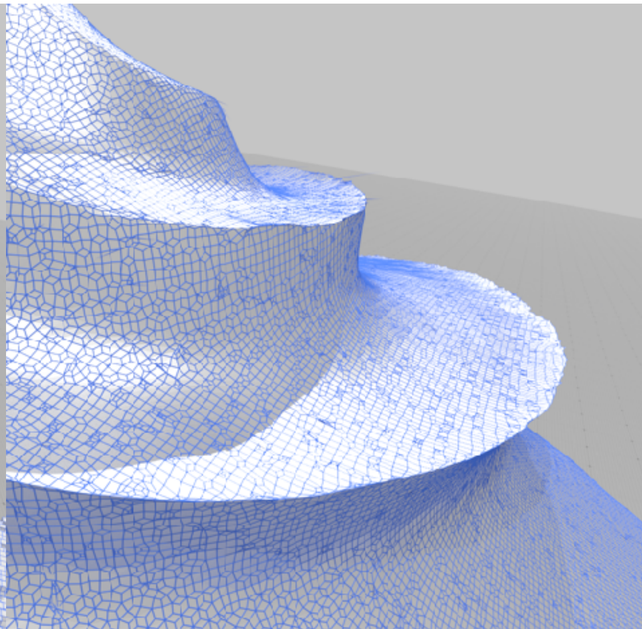




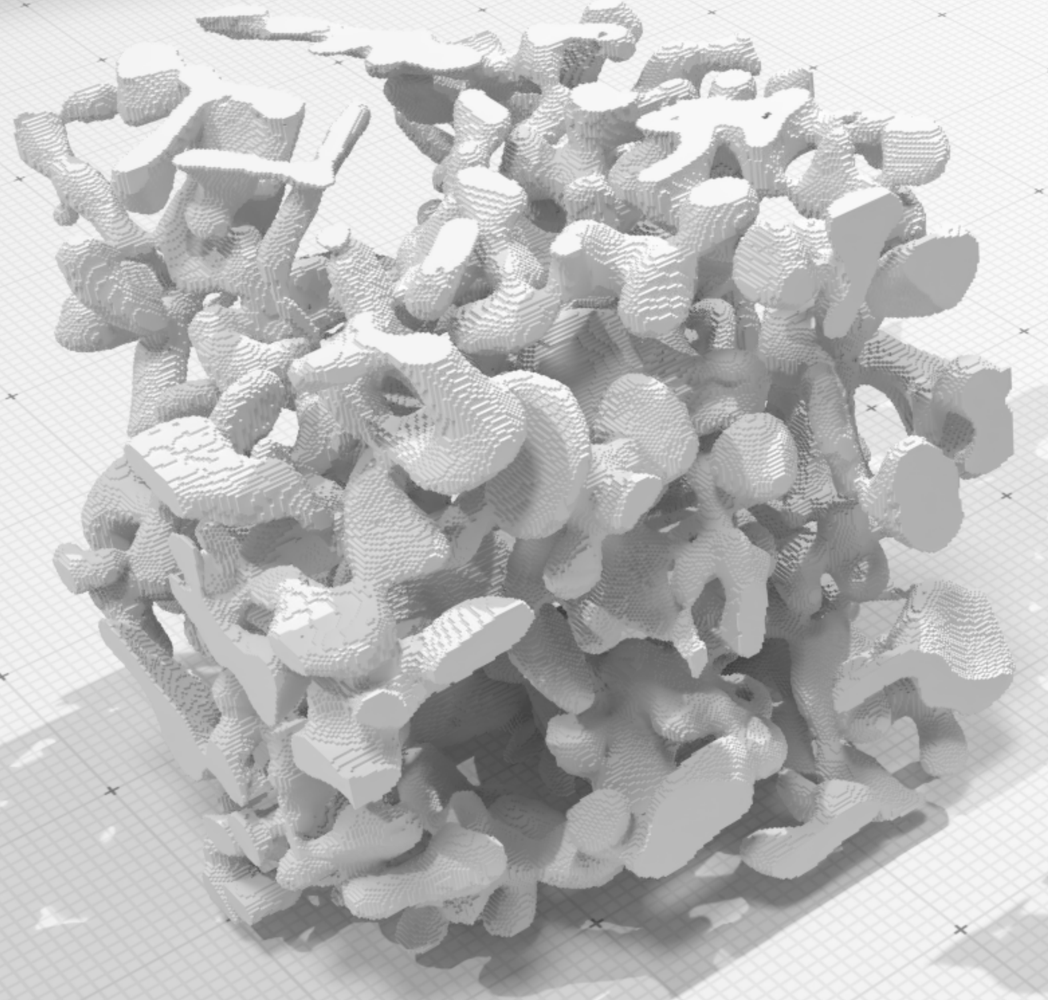


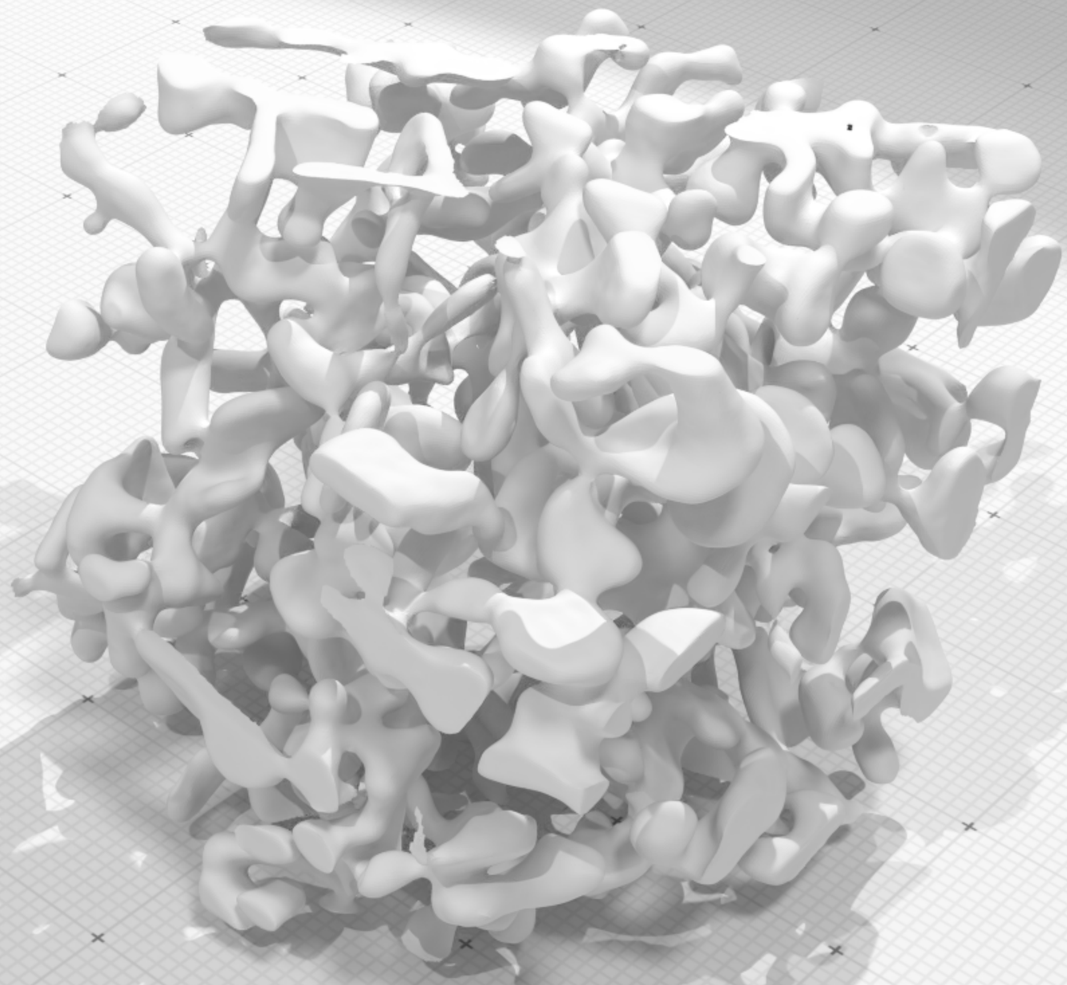




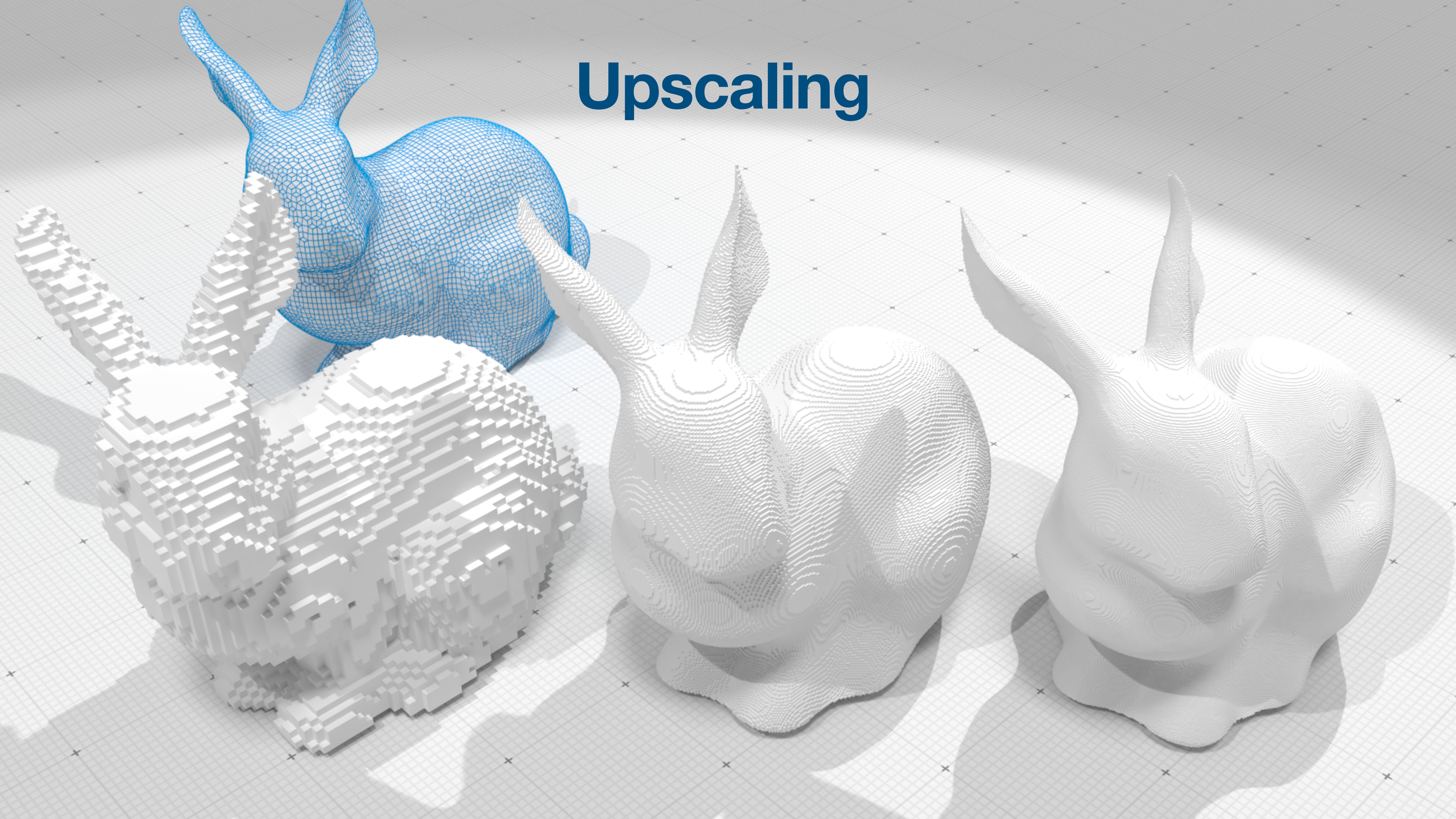


Piecewise smooth reconstruction

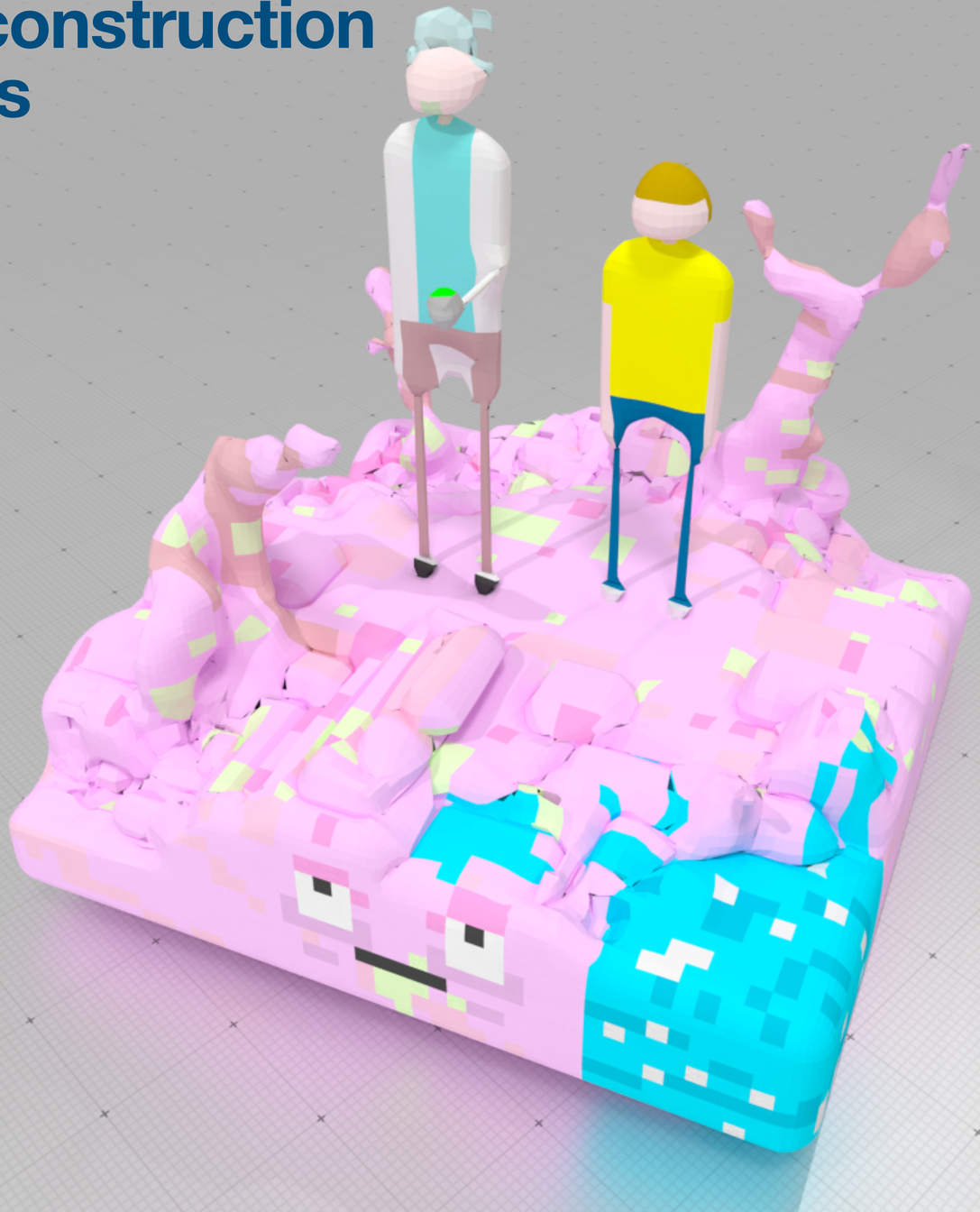




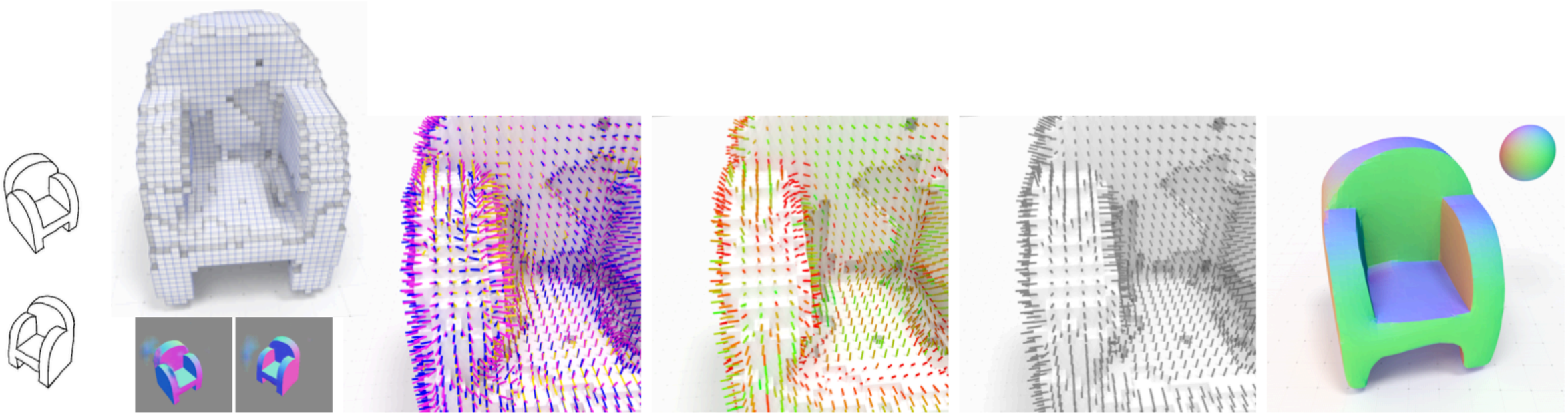
Upscaling



Piecewise smooth reconstruction of partitions



Example in geometric deep learning



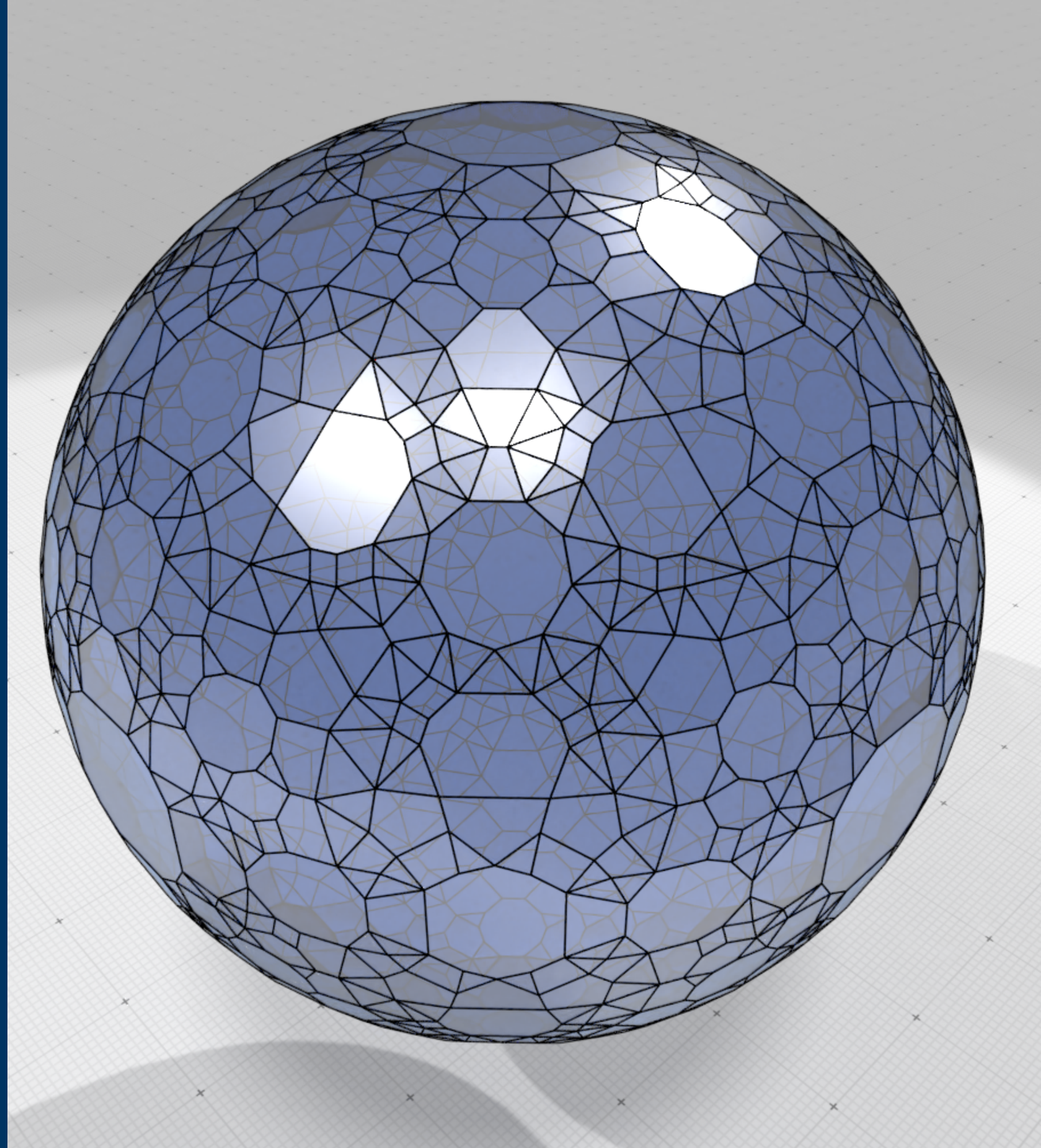
(a) Input (b) CNNs predictions (c) Candidate normals (d) Aggregated normals (e) Piecewise-smooth normals (f) Final surface

Deep learning

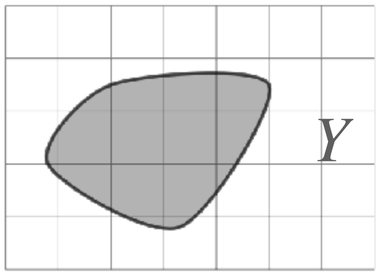
Digital geometry and calculus

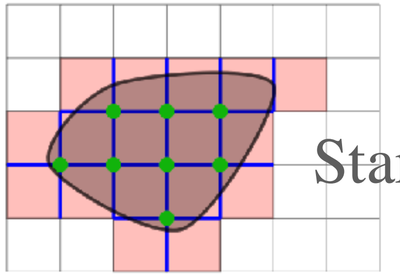
[Delanoy, Coeurjolly, L., Bousseau 19]

To conclude: some recent trends



Full convexity





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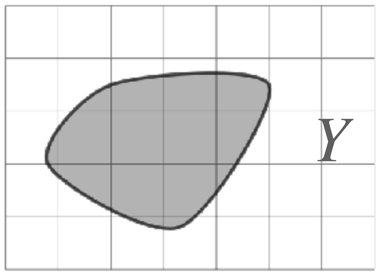

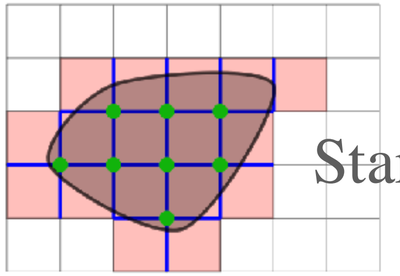

$X \subset \mathbb{Z}^d$ is **fully convex** iff

$\mathrm{Star}(X) = \mathrm{Star}(\mathrm{CvxH}(X))$

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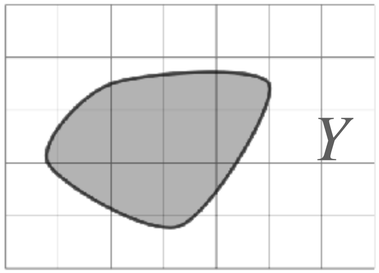

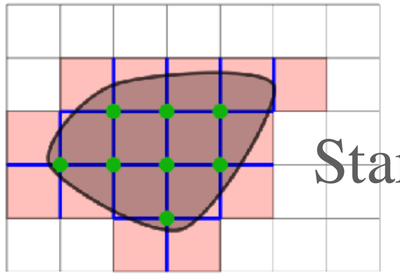

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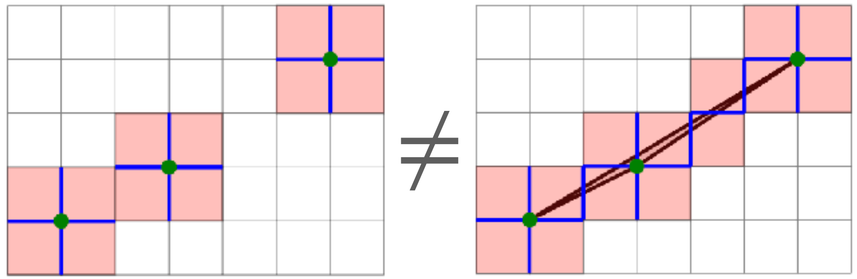

Digitally convex but not fully convex

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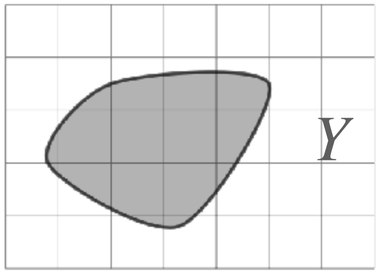

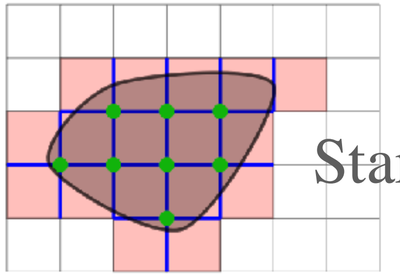

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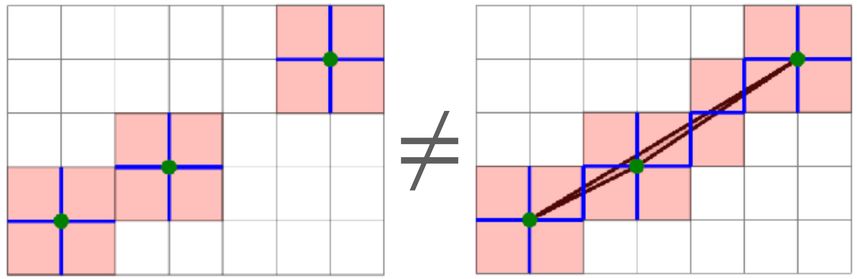
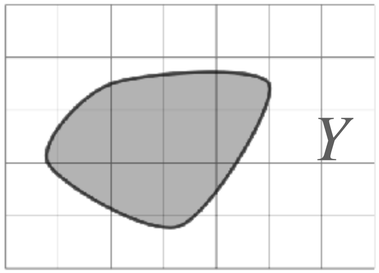
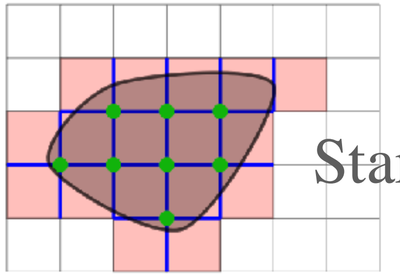


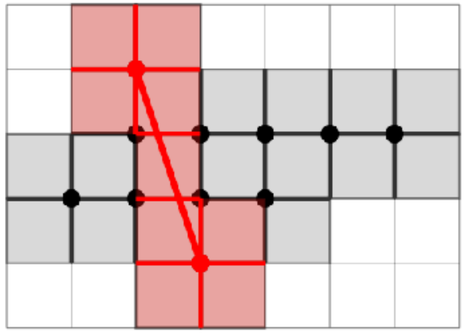
Digitally convex but not fully convex

Full convexity implies [L. 2021]
- digital convexity
- connectedness
- simple connectedness

Full convexity







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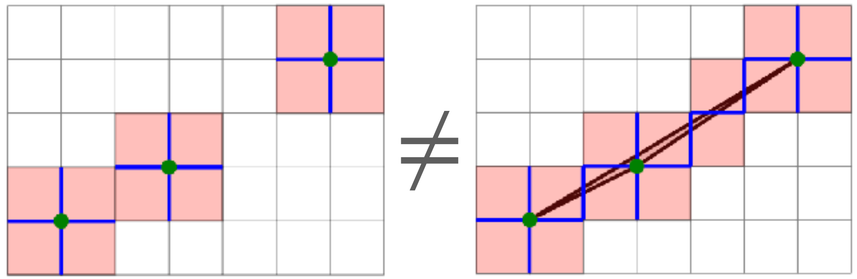

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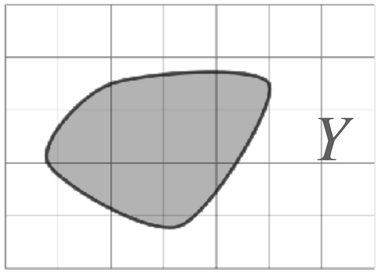

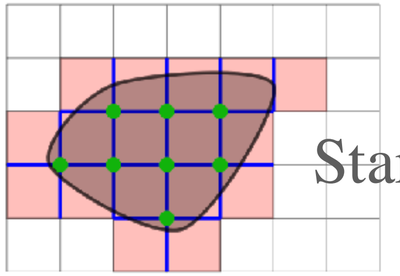

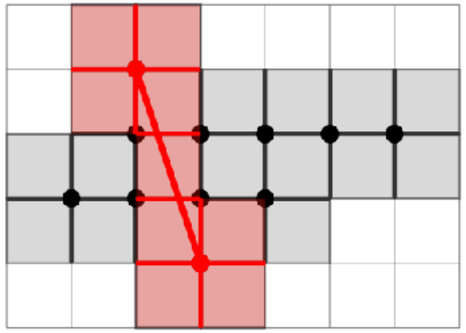

$\mathrm{Star}([pq]) \subset \mathrm{Star}(X)$

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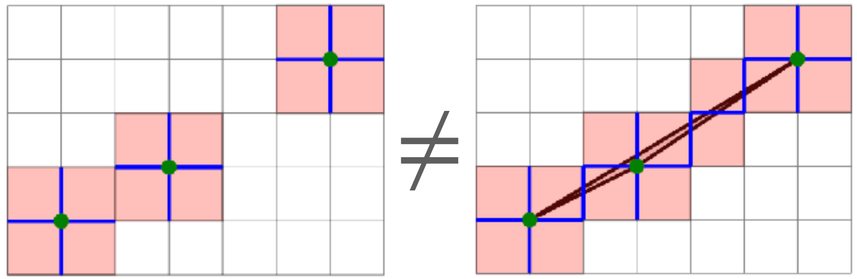

Digitally convex but not fully convex

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$X \subset \mathbb{Z}^d$ is P**-convex** iff
(i) $X = \mathrm{CvxH}(X) \cap \mathbb{Z}^d$,
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[Feschet, L. 24]

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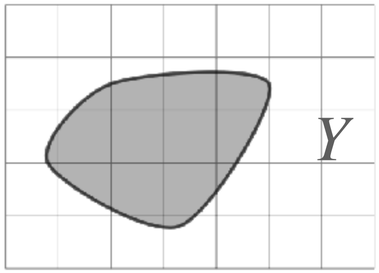
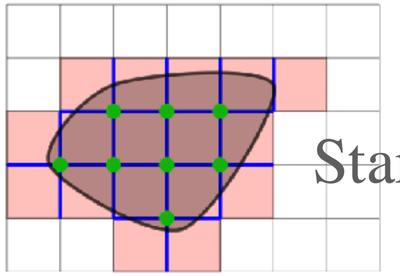
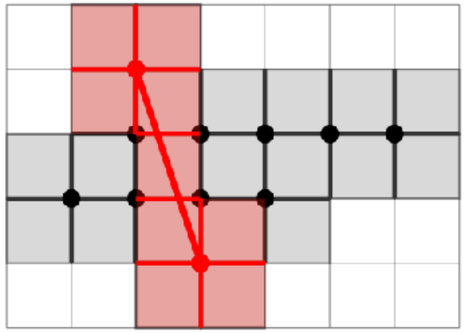
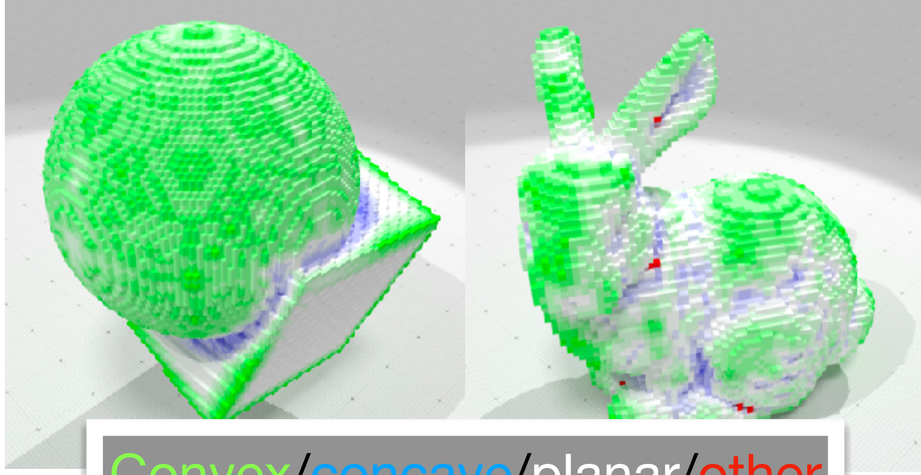
Full convexity







Star ([pq]) ⊂ Star (X)



Convex/concave/planar/other

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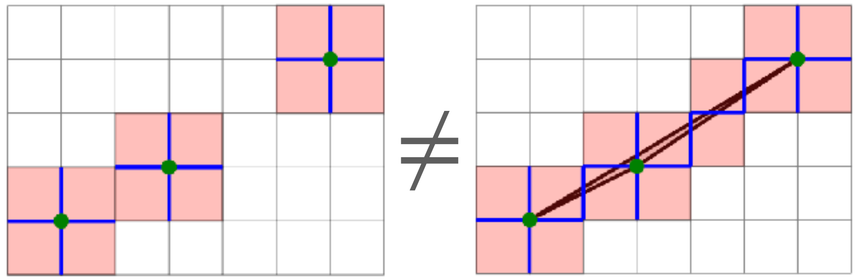

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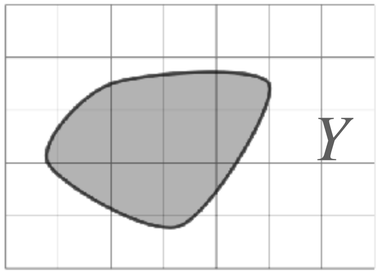

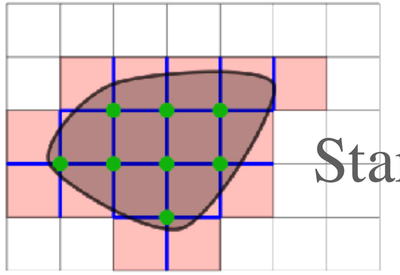

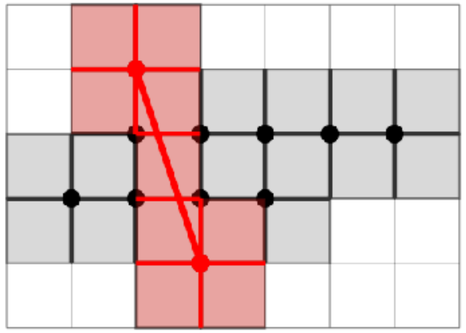

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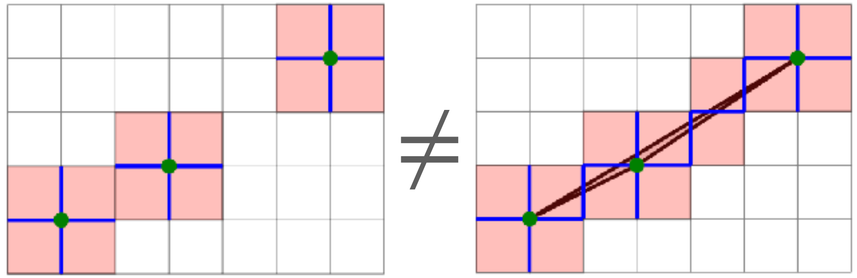

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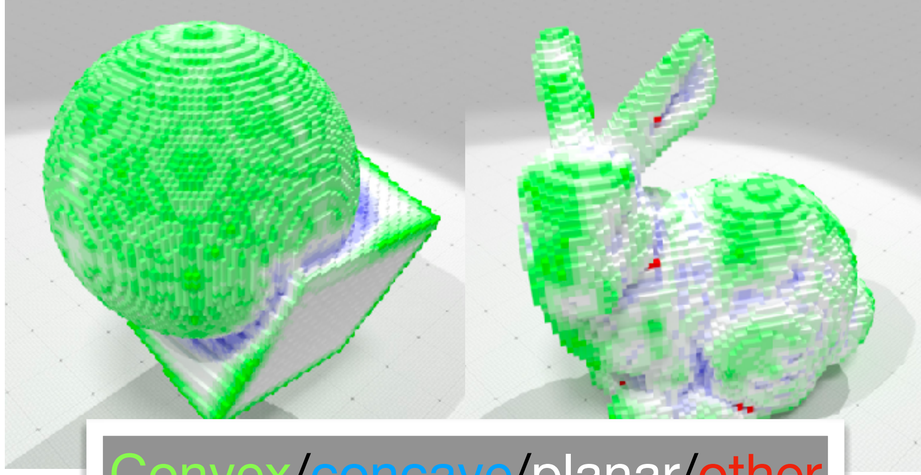
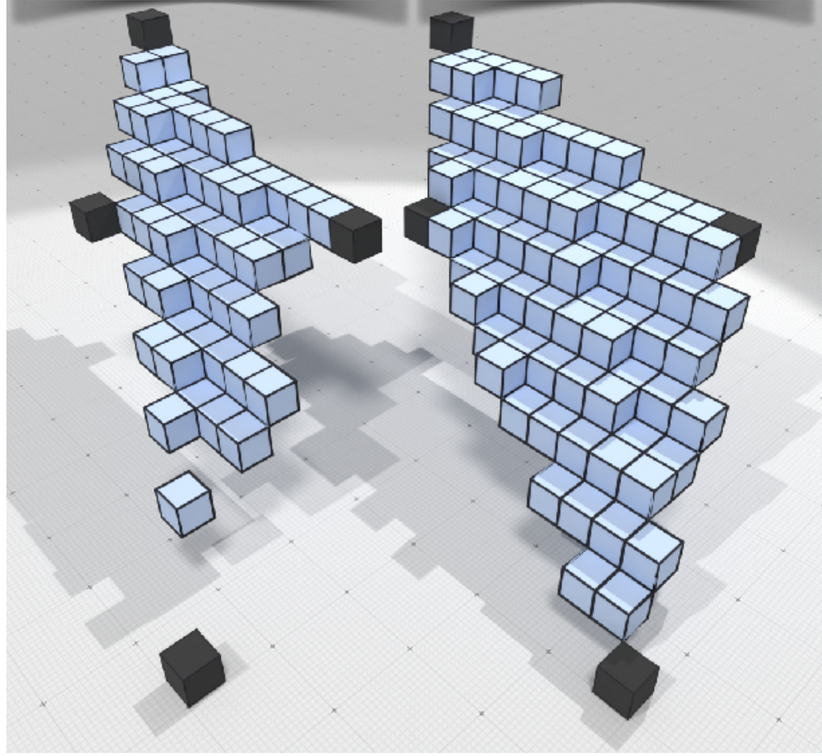


Convex/concave/planar/other

Digital convex hull

Fully convex envelope



Visibility = full sub-convexity

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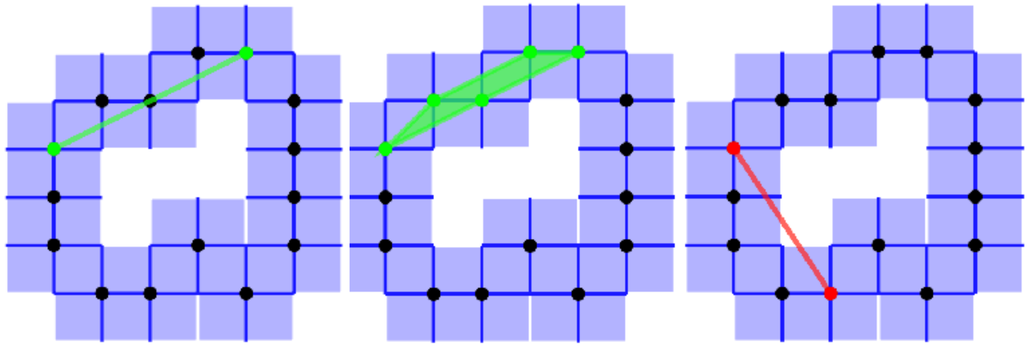

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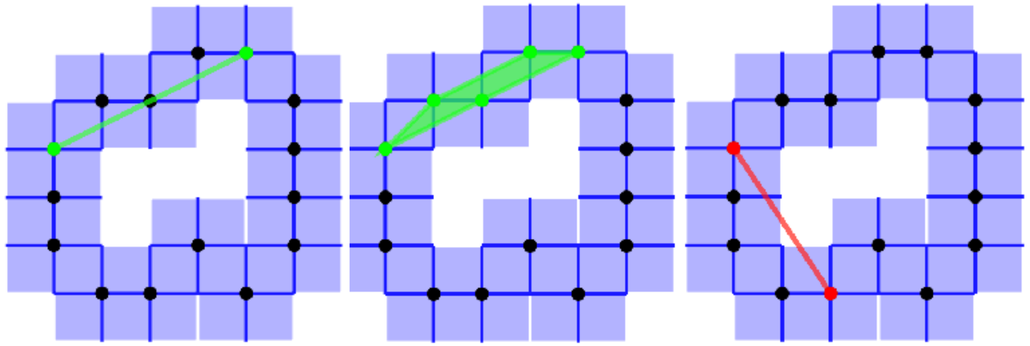

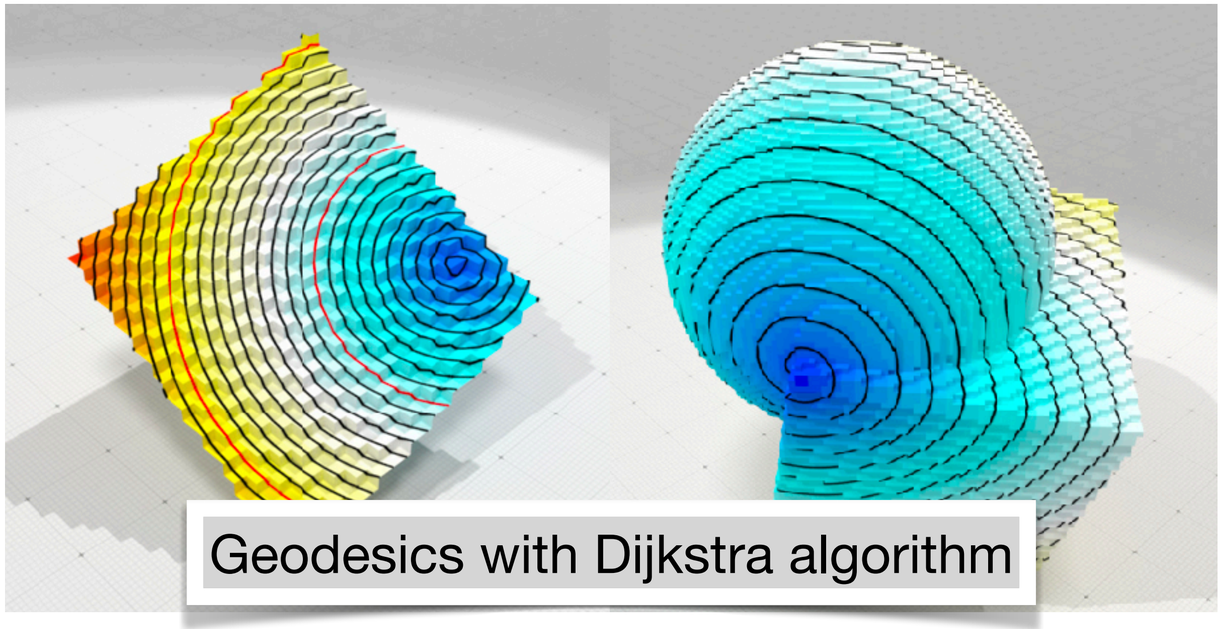

Geodesics with Dijkstra algorithm

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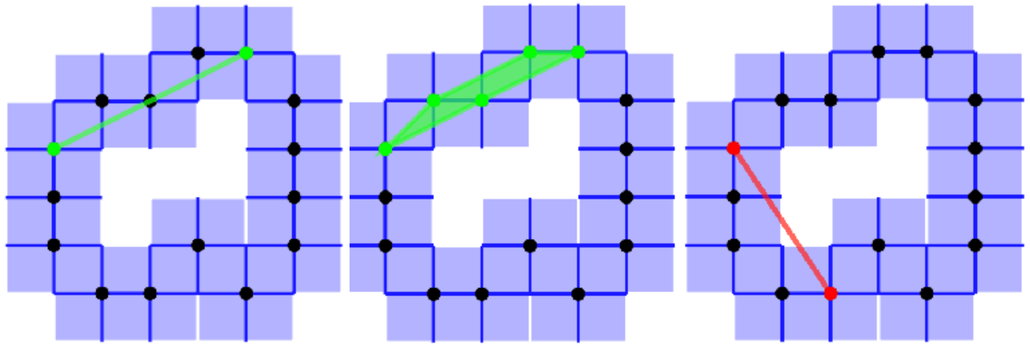
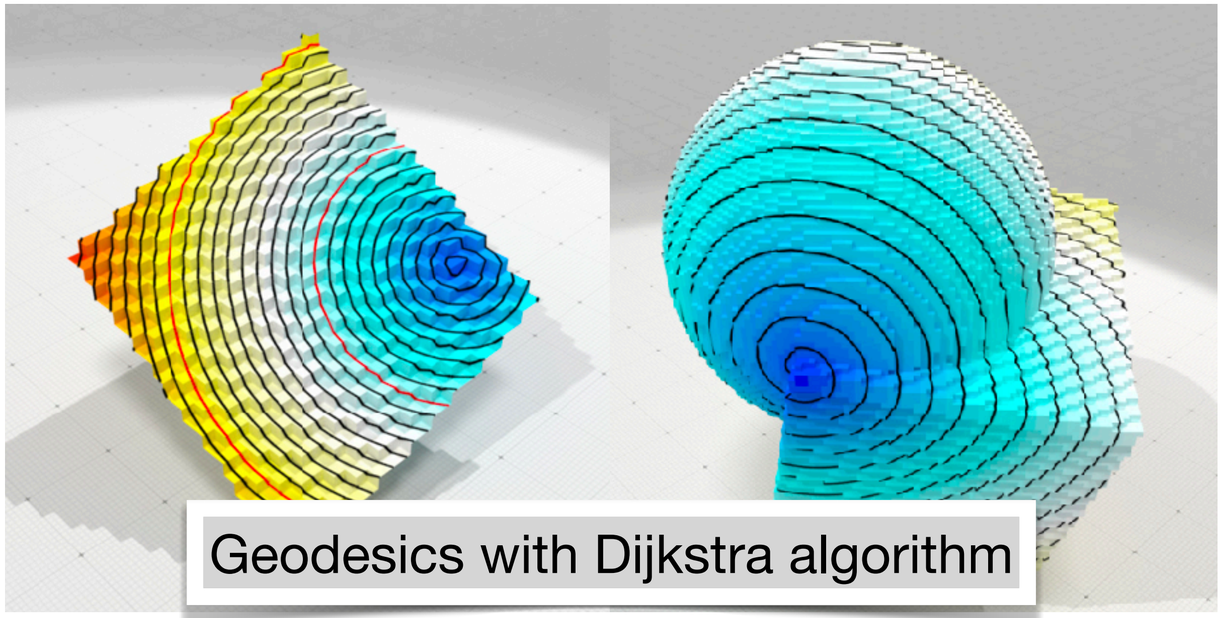
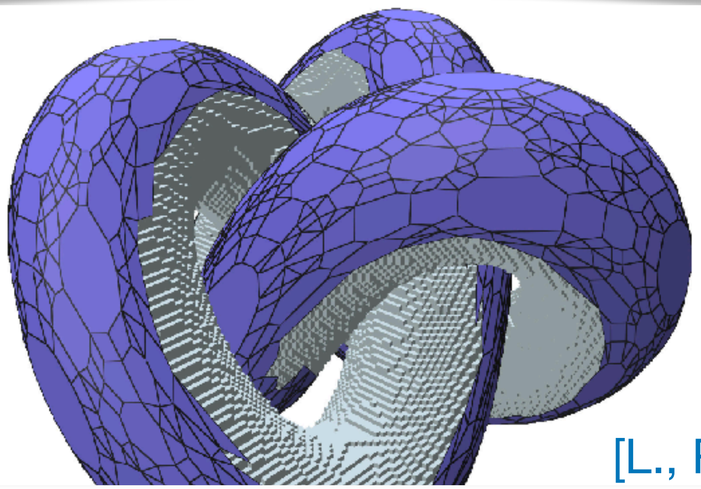




Geodesics with Dijkstra algorithm



[L., Roussillon 25]

Exact local convexity and concavities

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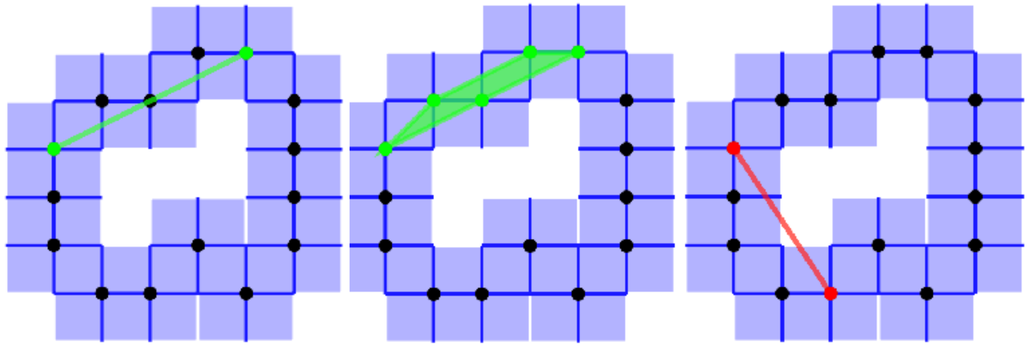
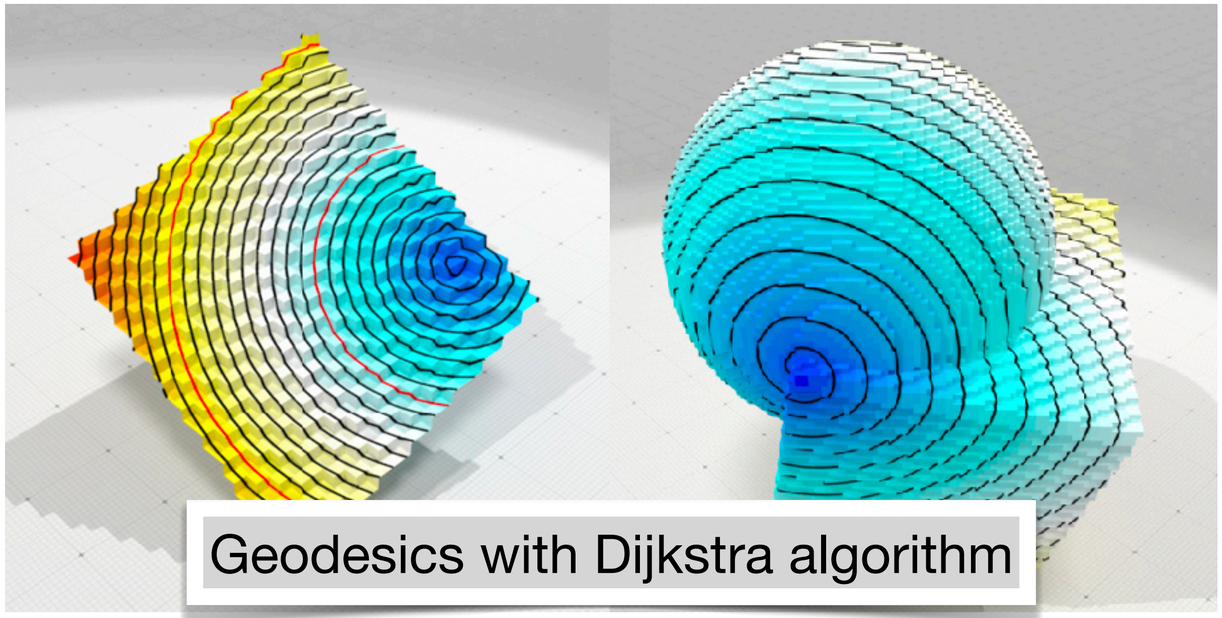
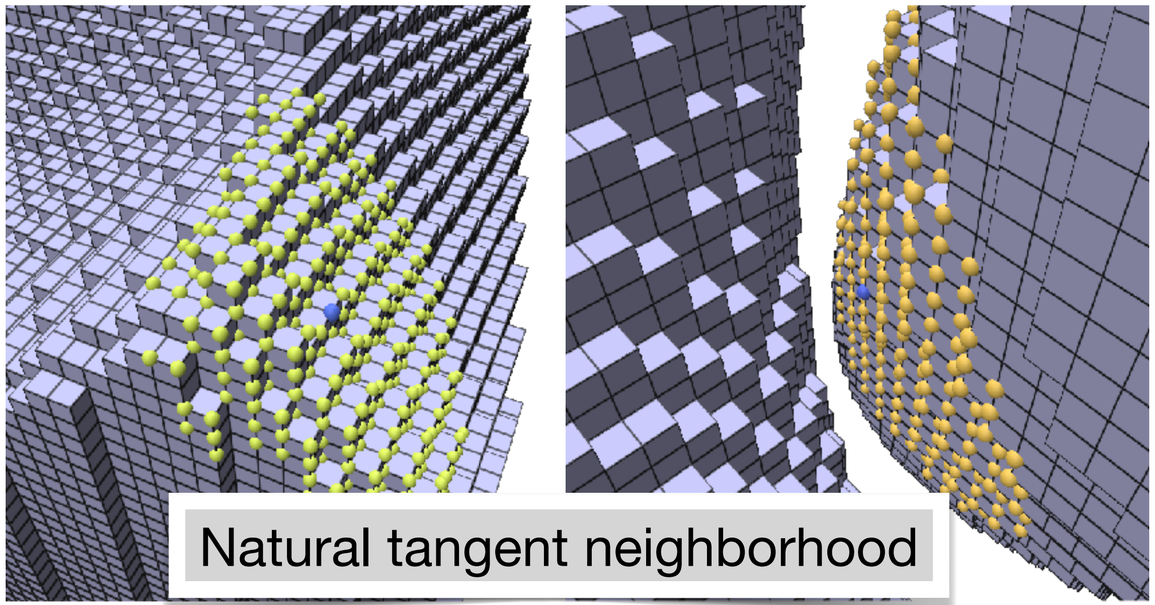
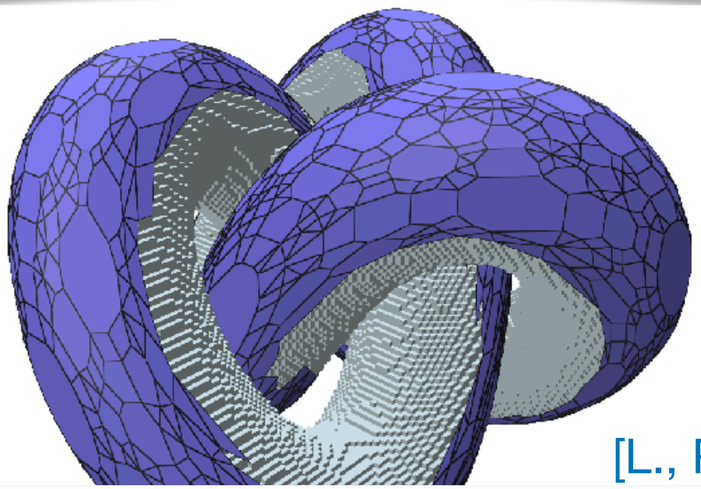




Geodesics with Dijkstra algorithm



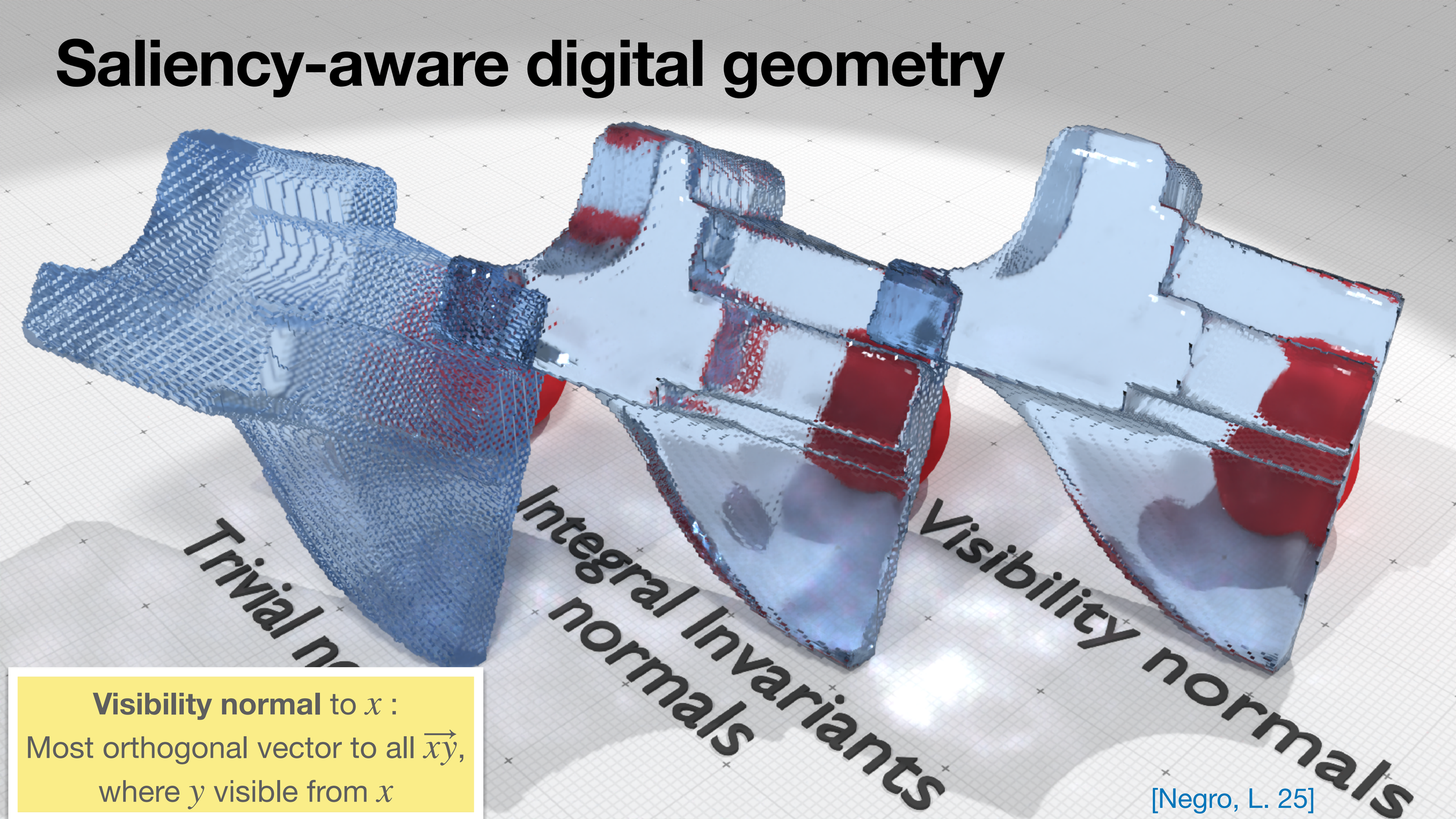
Natural tangent neighborhood



[L., Roussillon 25]

Exact local convexity and concavities

Saliency-aware digital geometry

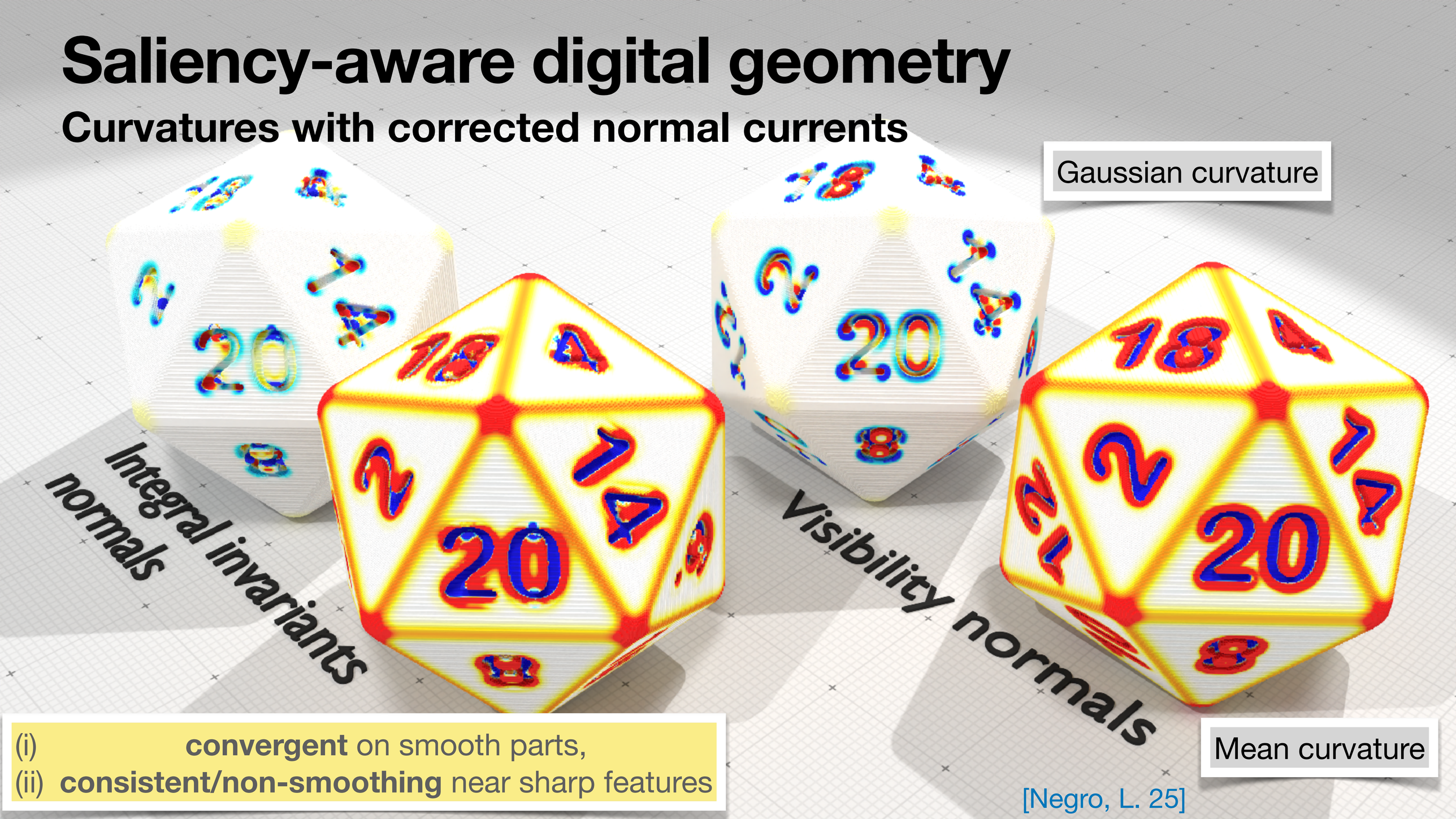


Visibility normal to x :
Most orthogonal vector to all $\overrightarrow{xy}$,
where y visible from x

[Negro, L. 25]

Saliency-aware digital geometry

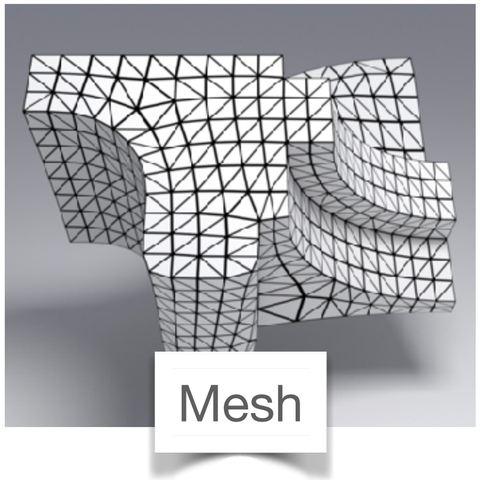
Curvatures with corrected normal currents



(i) **convergent** on smooth parts,
(ii) **consistent/non-smoothing** near sharp features

[Negro, L. 25]

What about AI ?



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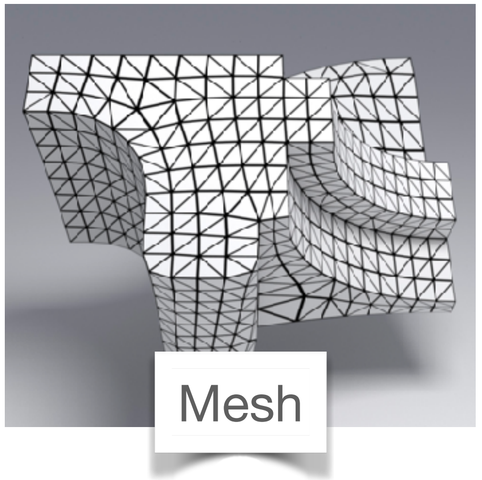

Geometry processing

mesh filtering, enhancement, repairing,
compression, meshing, reconstruction
mesh segmentation, indexation, retrieval,
measurements, quality-control,
rendering, prototyping, …

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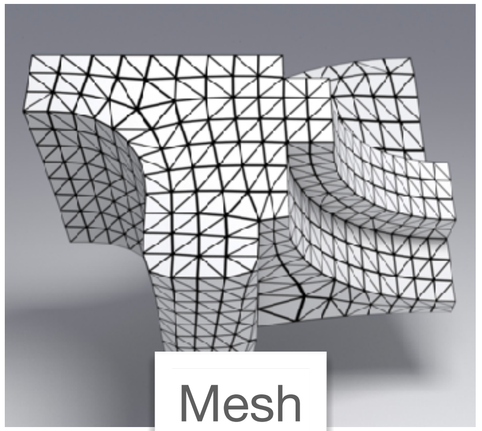

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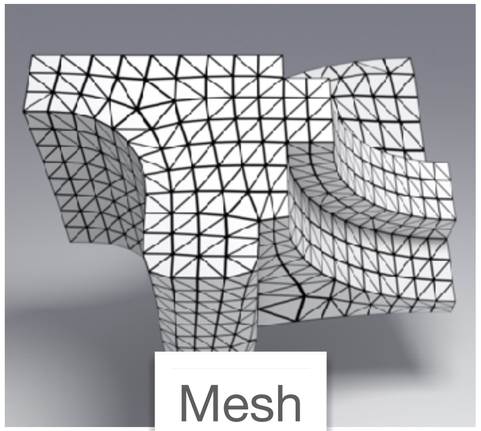

Mesh

Classical methods

Algo 1

Algo 2

Algo n

Geometry processing

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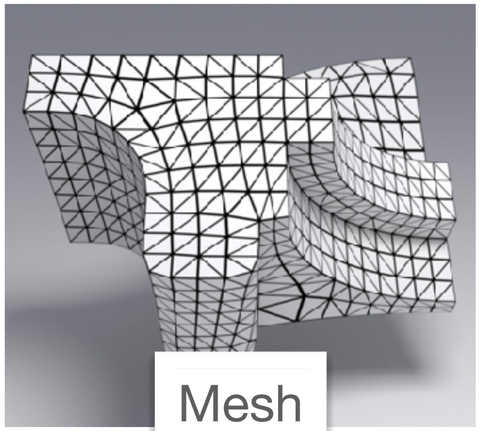

Mesh

Classical methods

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Algo 2

Algo n

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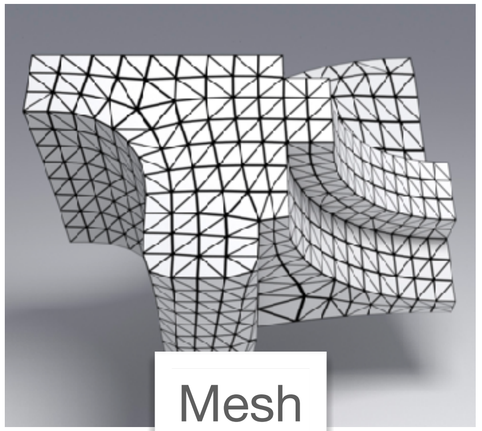

Mesh

Classical methods

Algo 1

Algo 2

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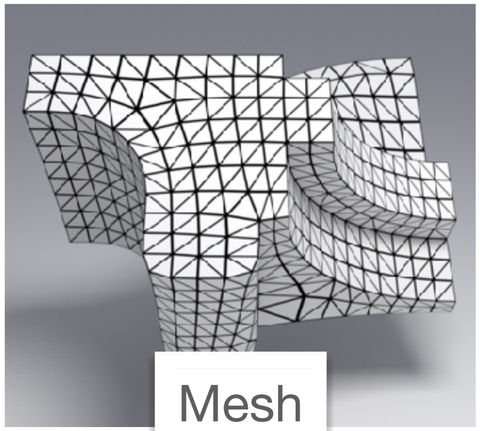

Mesh

Geometry
(normals, curvatures, saliencies, …)

Classical methods

Algo 1

Algo 2

Algo n

Geometry processing
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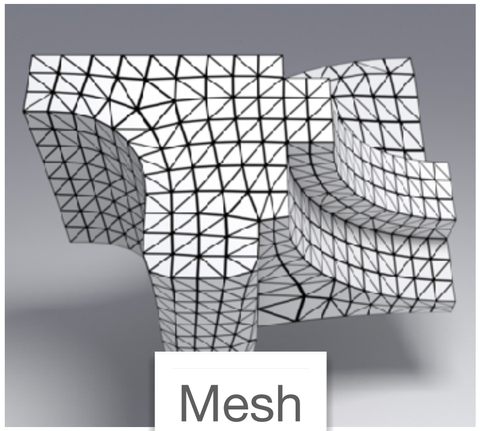

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(normals, curvatures, saliencies, …)

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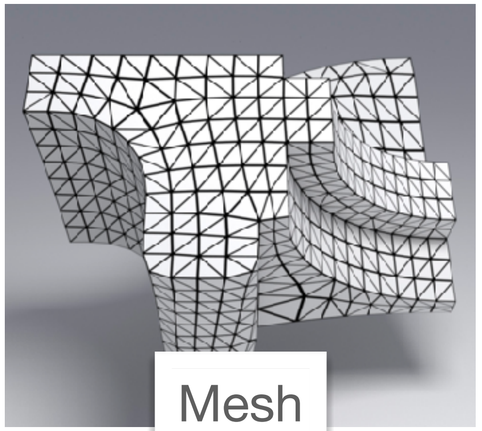

Mesh

Points cloud

Geometry
(normals, curvatures, saliencies, …)

Classical methods

Algo 1

Algo 2

Algo n

Geometry processing
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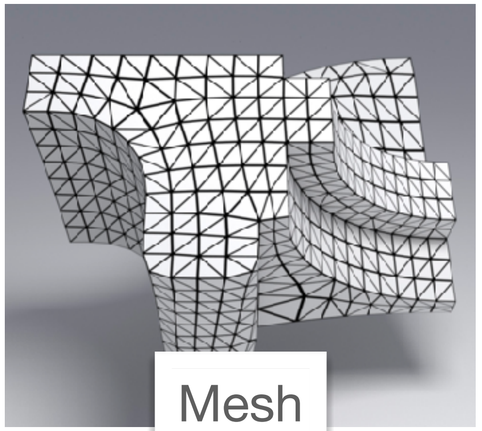

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Points cloud

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(normals, curvatures, saliencies, …)

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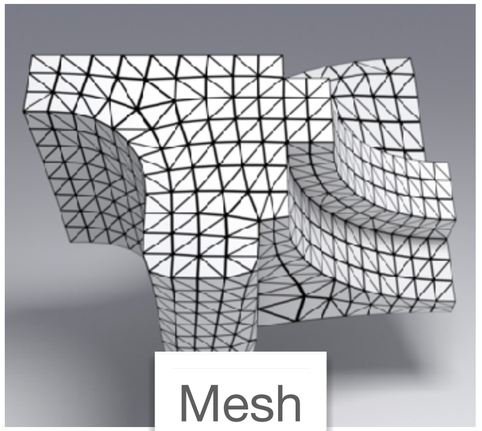

Mesh

Points cloud

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(normals, curvatures, saliencies, …)

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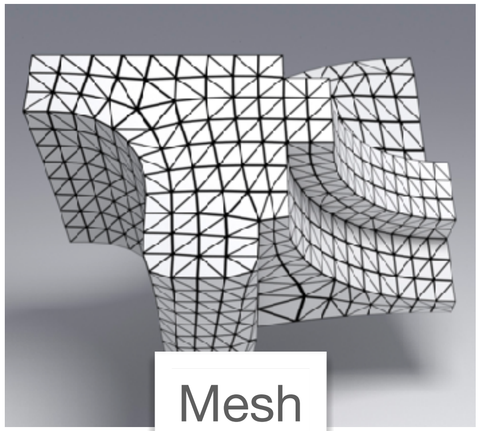

Mesh

Points cloud

Geometry
(normals, curvatures, saliencies, …)

Classical methods

Algo 1

Algo 2

Algo n

Geometric deep learning

Algo 1

Algo 2

Algo n

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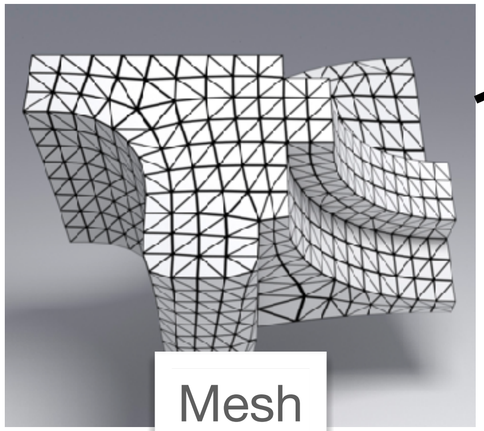

Mesh

Points cloud

Geometric deep learning

- Algo 1
- Algo 2
- ⋮
- Algo n

Classical methods

- Algo 1
- Algo 2
- ⋮
- Algo n

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(normals, curvatures, saliencies, …)

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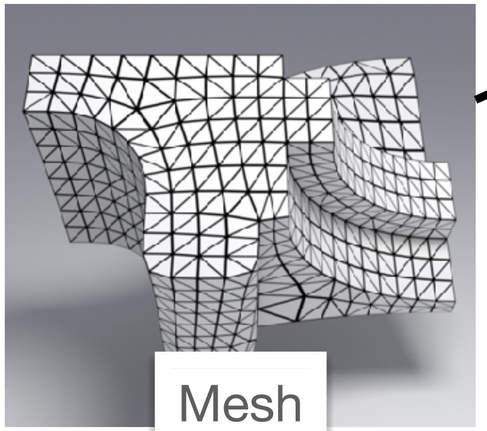

Mesh

Points cloud

Geometric deep learning

- Algo 1
- Algo 2
- ⋮
- Algo n

Classical methods

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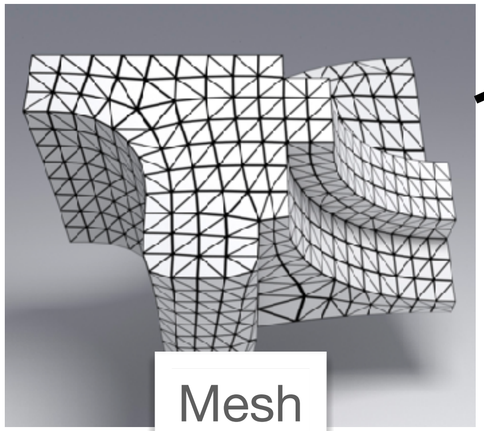

Mesh

Points cloud

Geometric deep learning

- Algo 1
- Algo 2
- ⋮
- Algo n

Classical methods

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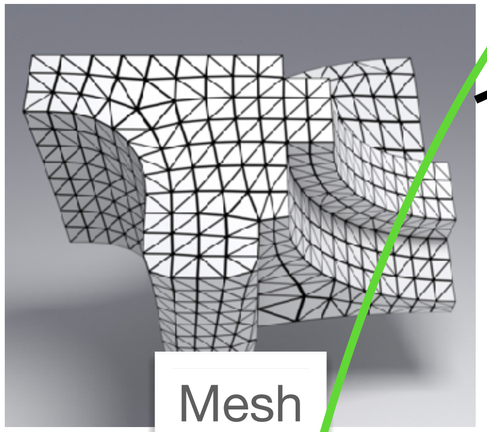

Mesh

Points cloud

Geometric deep learning

- Algo 1
- Algo 2
- ⋮
- Algo n

Classical methods

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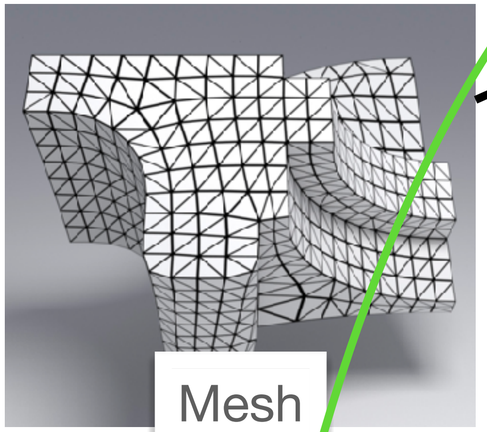
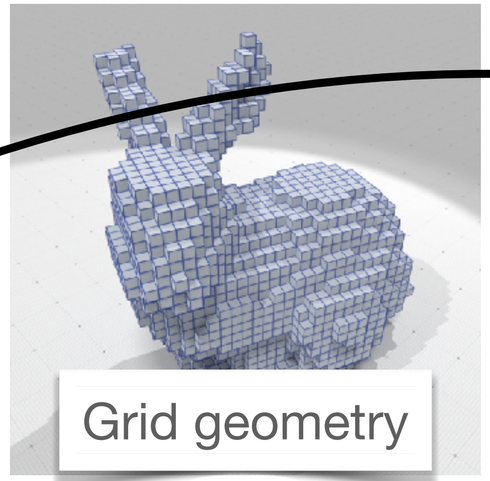


Mesh

Points cloud



Grid geometry

Geometric deep learning

- Algo 1
- Algo 2
- ⋮
- Algo n

Geometry
(normals, curvatures, saliencies, …)

Classical methods

- Algo 1
- Algo 2
- ⋮
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Geometry processing
mesh filtering, enhancement, repairing, compression, meshing, reconstruction mesh segmentation, indexation, retrieval, measurements, quality-control, rendering, prototyping, …

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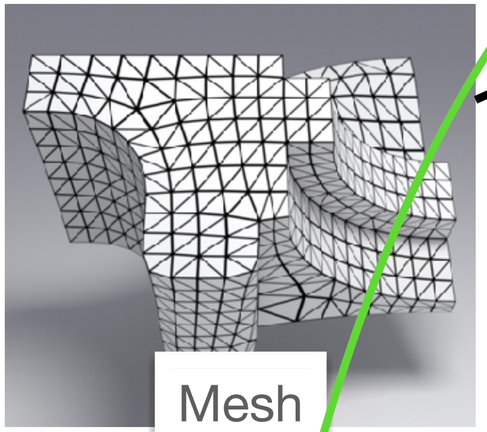
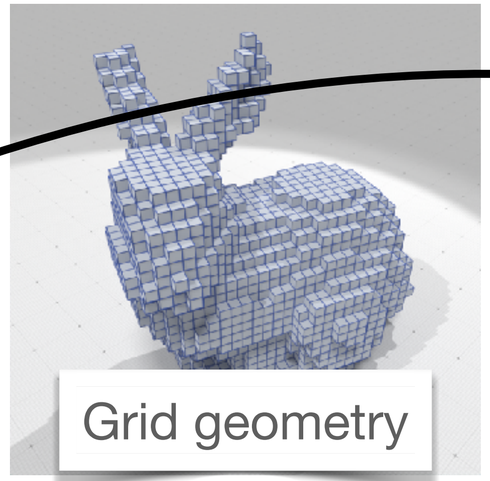


Mesh

Points cloud



Grid geometry

Geometric deep learning

- Algo 1
- Algo 2
- ⋮
- Algo n

Geometry
(normals, curvatures, saliencies, …)

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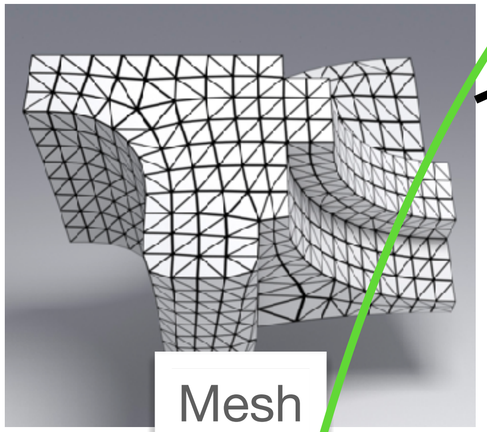
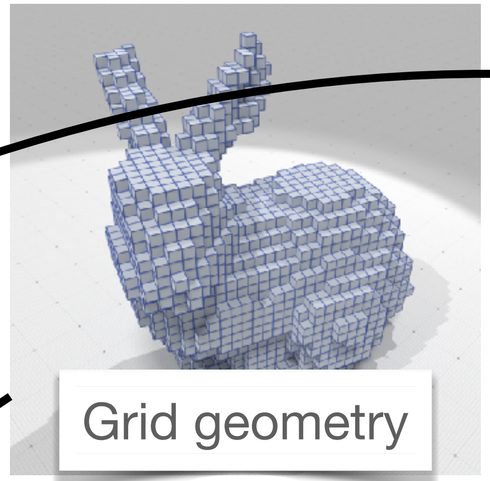


Mesh

Points cloud



Grid geometry

Geometric deep learning

- Algo 1
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(normals, curvatures, saliencies, …)

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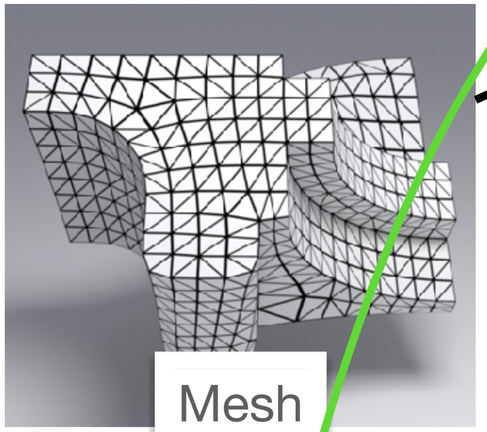
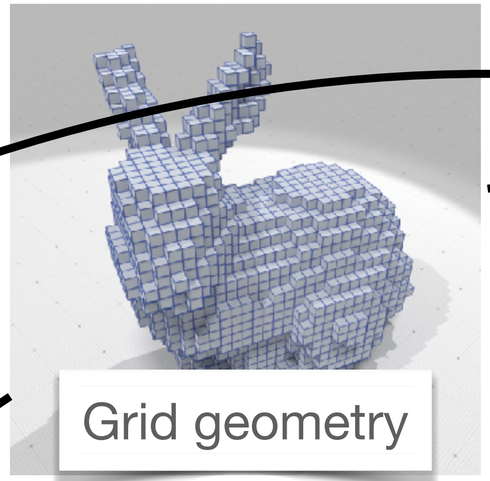


Mesh

Points cloud



Grid geometry

Geometric deep learning

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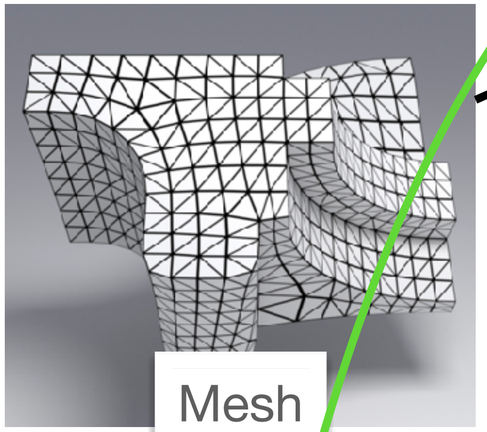

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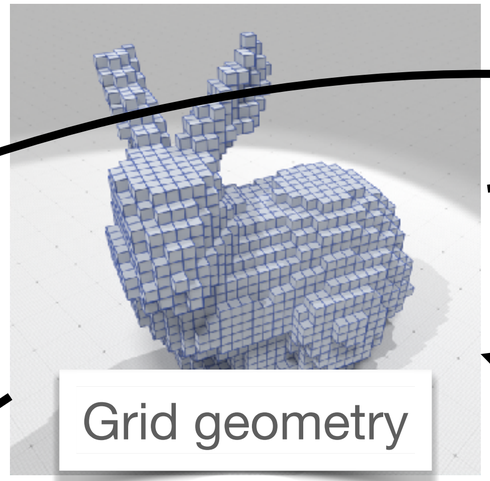
Points cloud



Grid geometry

Geometric deep learning

- Algo 1
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- ⋮
- Algo n

Classical methods

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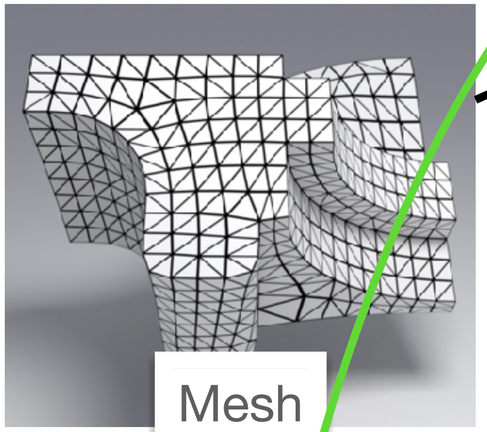
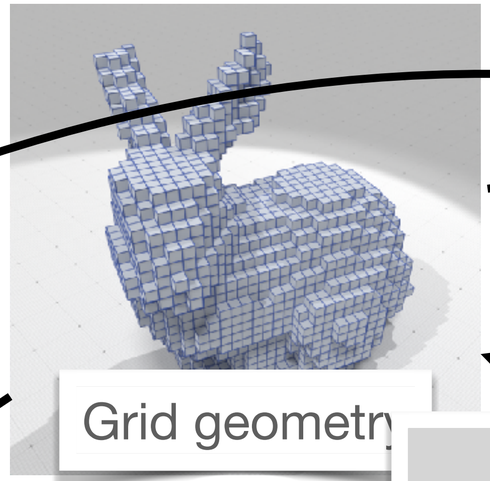


Mesh

Points cloud



Grid geometry

Geometric deep learning

Algo 1

Algo 2

Algo n

Sparse voxel grids
[NVIDIA nanoVDB & fVDB]

Classical methods

Algo 1

Algo 2

Algo n

Geometry
(normals, curvatures, saliencies, …)

Geometry processing
mesh filtering, enhancement, repairing, compression, meshing, reconstruction
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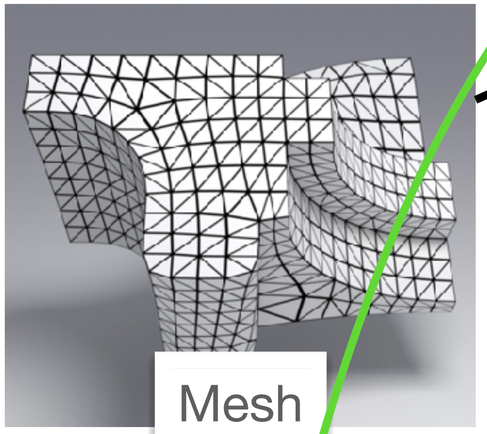
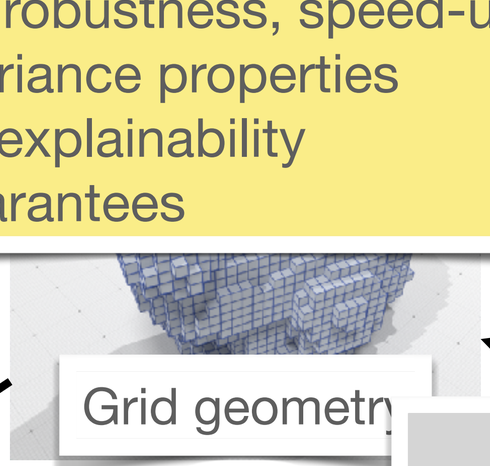
What about AI ?

Incorporating **geometry** in **SVG** could
- Increase robustness, speed-up training
- Add invariance properties
- Improve explainability
- Offer guarantees



Mesh

Points cloud



Grid geometry

Geometric deep learning

Algo 1

Algo 2

Algo n

Sparse voxel grids
[NVIDIA nanoVDB & fVDB]

Classical methods

Algo 1

Algo 2

Algo n

Geometry
(normals, curvatures, saliencies, …)

Geometry processing
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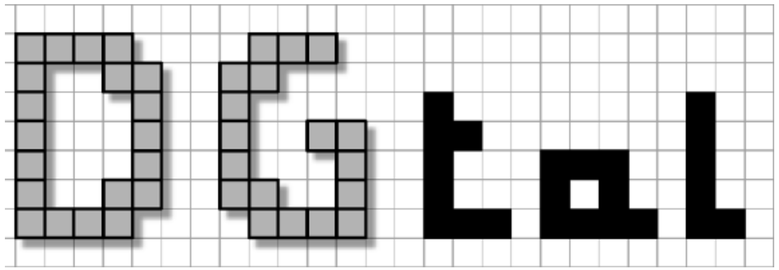
Conclusion

Geometry in the regular grid is a sound alternative to mesh or point cloud geometry.

- Topology and geometry inference
- Calculus and geometry processing
- Fast, exact, and stable numerical algorithms … in $\mathbb{Z}^d$



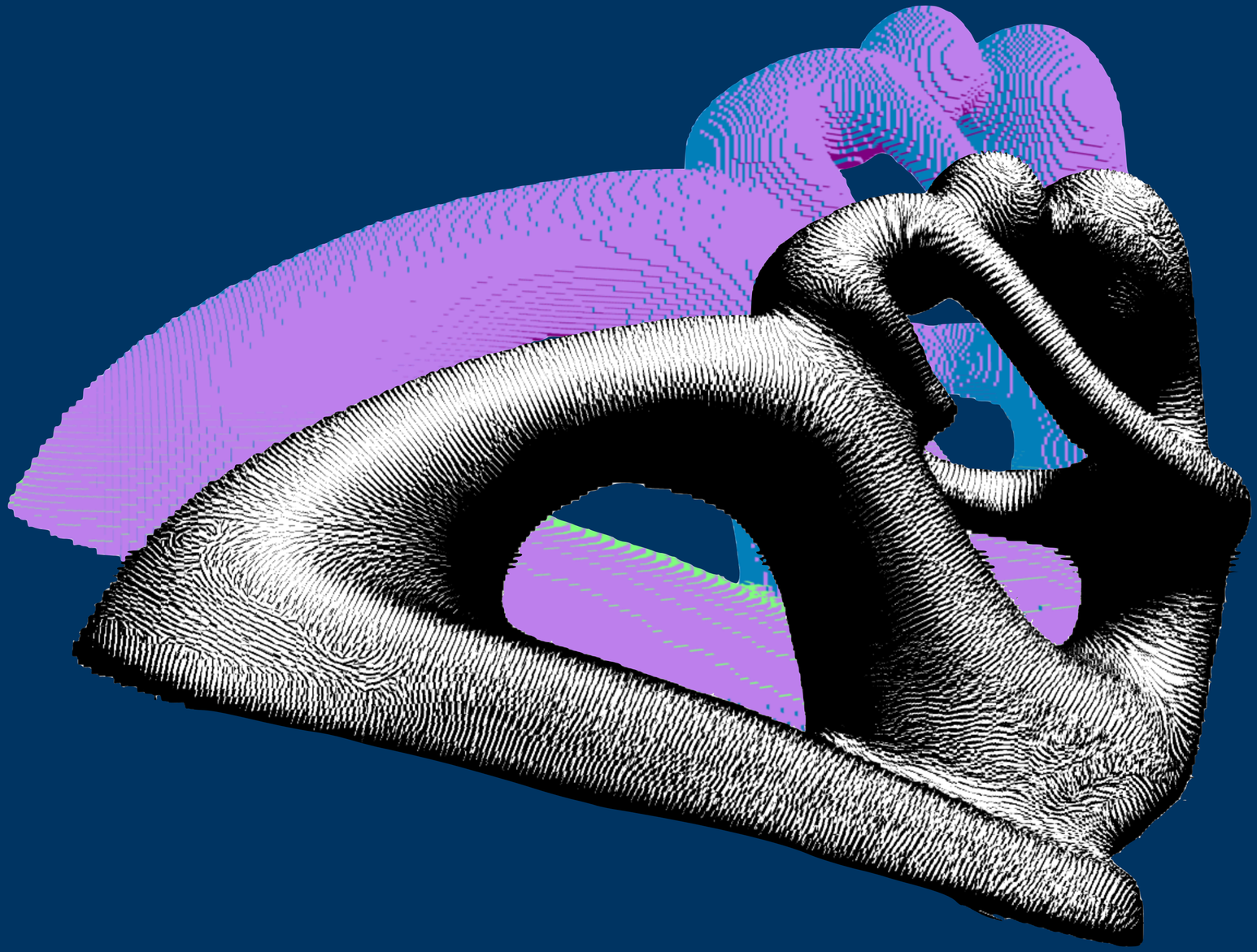
dgtal.org

Discord

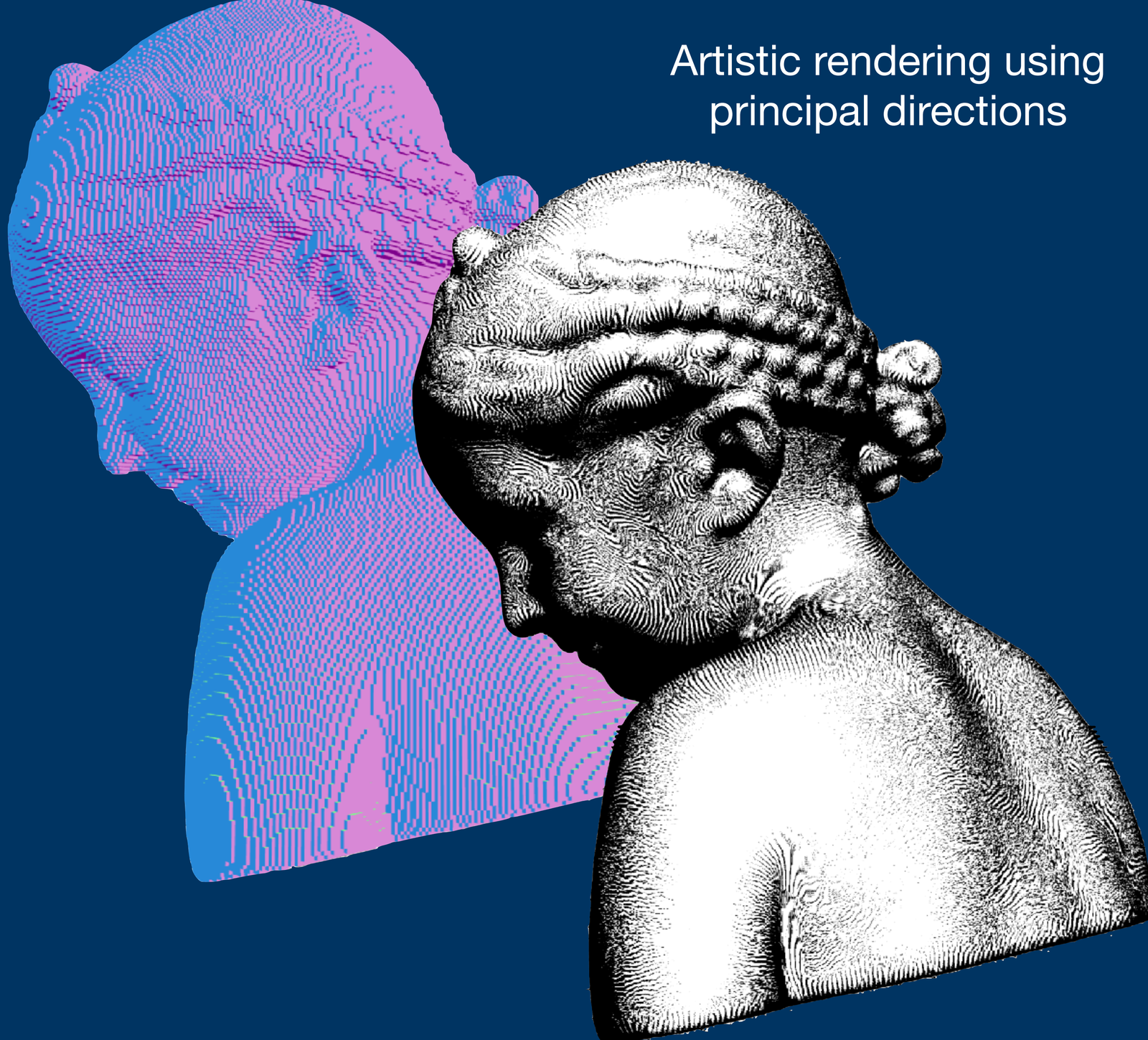
https://discord.gg/PtCk6vtS

Thank you !

Artistic rendering using
principal directions



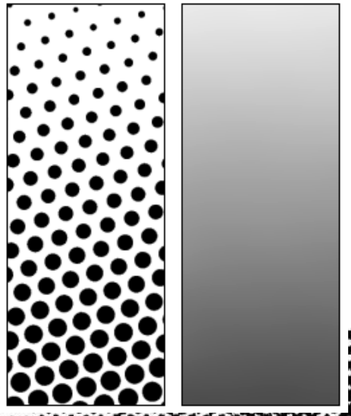
Thank you !



Artistic rendering using
principal directions

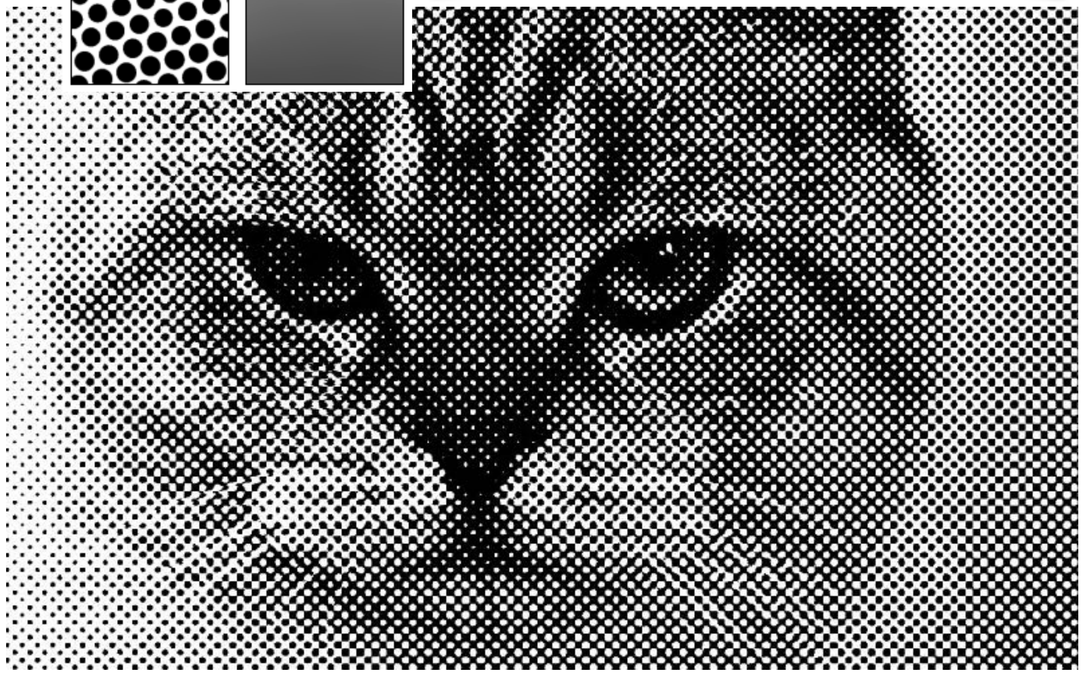
Non-photorealistic rendering

Artistic halftoning, copper engraving



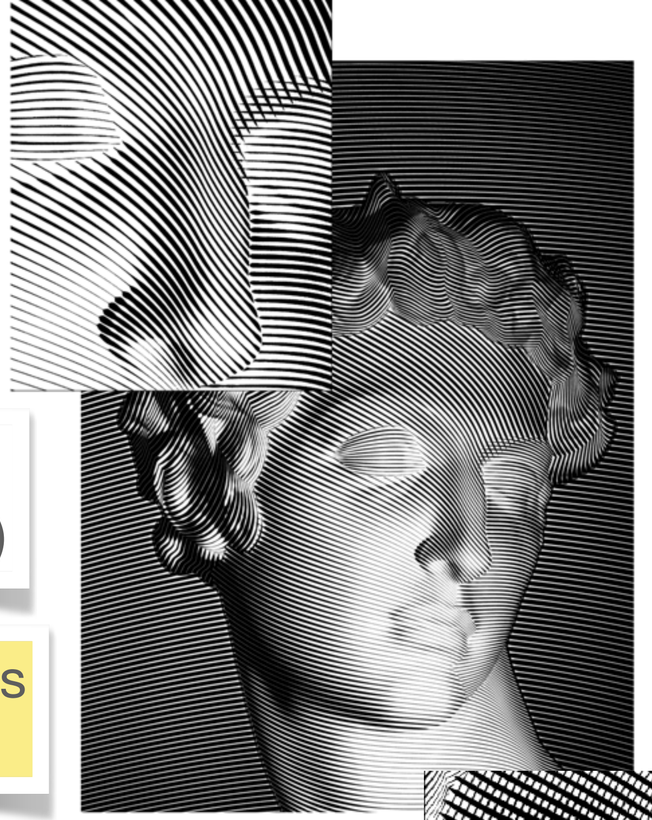
Artistic halftoning
(standard printing)

Use brightness, normals



Copper engraving
(e.g. old illustrations)

Use brightness, normals
principal directions

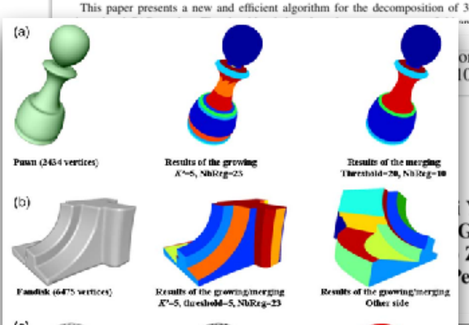


[Ostromoukhov 99]

Why curvatures ?

Why curvatures ?

Mesh segmentation



ELSEVIER
Computer-Aided Design 37 (2005) 975–987
COMPUTER-AIDED DESIGN

A new CAD mesh segmentation method
curvature tensor analysis

Guillaume Lavoué*, Florent Dupont, Atilla E
LIRIS FRE 2672 CNRS, 43, Bd du 11 novembre, 69622 Villeurbanne Ced

Received 17 April 2004; received in revised form 7 September 2004; accepted 2

Abstract

This paper presents a new and efficient algorithm for the decomposition of 3D arbitrary tria

IEEE TRANSACTIONS ON PATTERN ANALYSIS AND MACHINE INTELLIGENCE, VOL. 29, NO. 12, DECEMBER 2007 2195

Three-Dimensional Surface Mesh Segmentation Using Curvedness-Based Region Growing Approach

Anupama Jagannathan, *Member, IEEE*, and Eric L. Miller, *Senior Member, IEEE*

Abstract—A new parameter-free graph-morphology-based segmentation algorithm is proposed to address the problem of partitioning a 3D triangular mesh into disjoint submeshes that correspond to the physical parts of the underlying object. Curvedness, which is a rotation

omput (2005) 21: 659–668
1007/s00371-005-0319-x
ORIGINAL ARTICLE

Yamauchi
Gumhold
Zayer
eter Seidel

Mesh segmentation driven by Gaussian curvature

Why curvatures ?

EUROGRAPHICS 2001 / A. Chalmers and T.-M. Rhyne
(Guest Editors)

Volume 20 (2001), Num

Detection of Salient Curvature
on Polygonal Surfaces

Kouki Watanabe † and Alexander G. Bely

The University of Aizu, Aizu-Wakamatsu City, Fukushima 965-

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SPECIAL SECTION ON BIG DATA LEARNING

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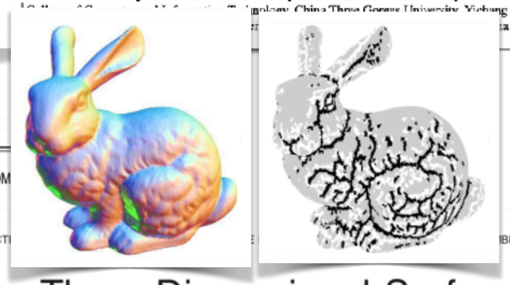

Mesh segmentation

ELSEVIER

Computer-Aided Design 37 (2005) 975–987

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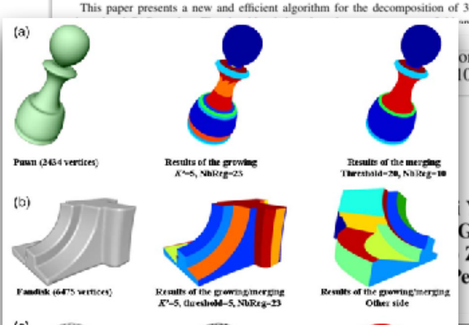

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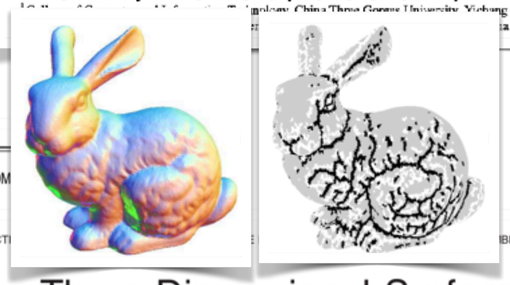

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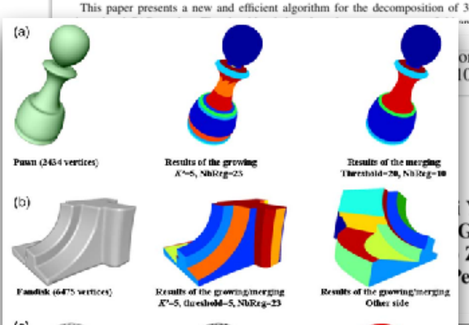

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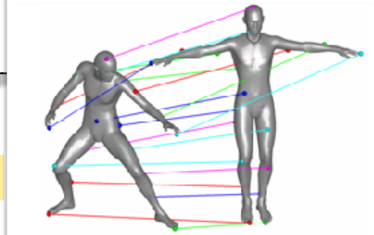

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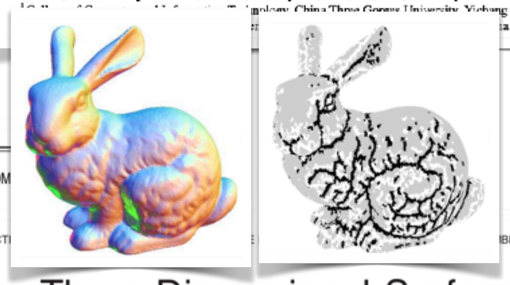

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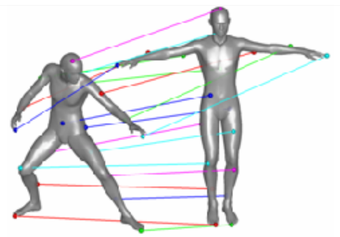

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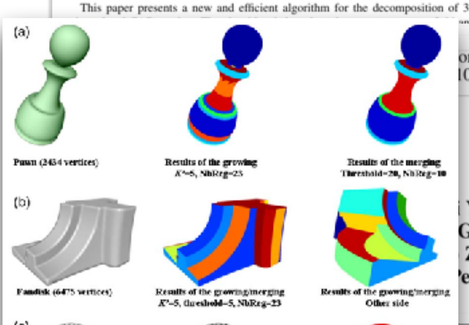

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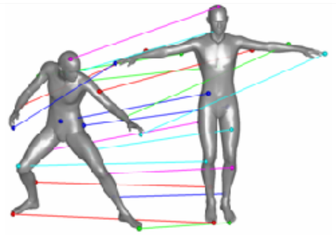

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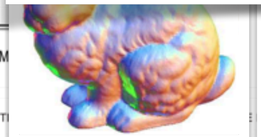

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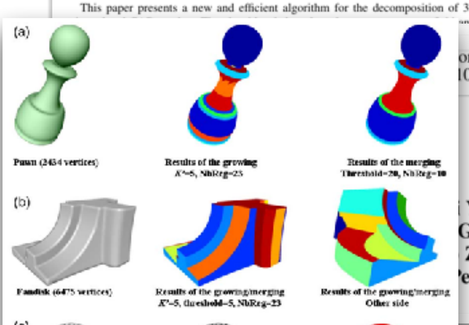

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ϵ positions error in $O(h)$

But η normals error in $O(1)$

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Same method works for point clouds with normals

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Computation is fast and simple since all formula are linear (in $\mathbb{R}^3 \times \mathbb{S}^2$)

Corrected Normal Currents

Interpolated CNC measures for triangle τ, positions $(\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k)$, normals $(\mathbf{u}_i, \mathbf{u}_j, \mathbf{u}_k)$

- *area measure:*

$$\mu^A_{\mathbf{u}}(\tau) = \int_\tau \Gamma^*_{\mathbf{u}} \omega^A = \frac{1}{2}\langle \bar{\mathbf{u}} \mid (\mathbf{x}_j - \mathbf{x}_i) \times (\mathbf{x}_k - \mathbf{x}_i) \rangle$$

- **m***ean curvature measure:*

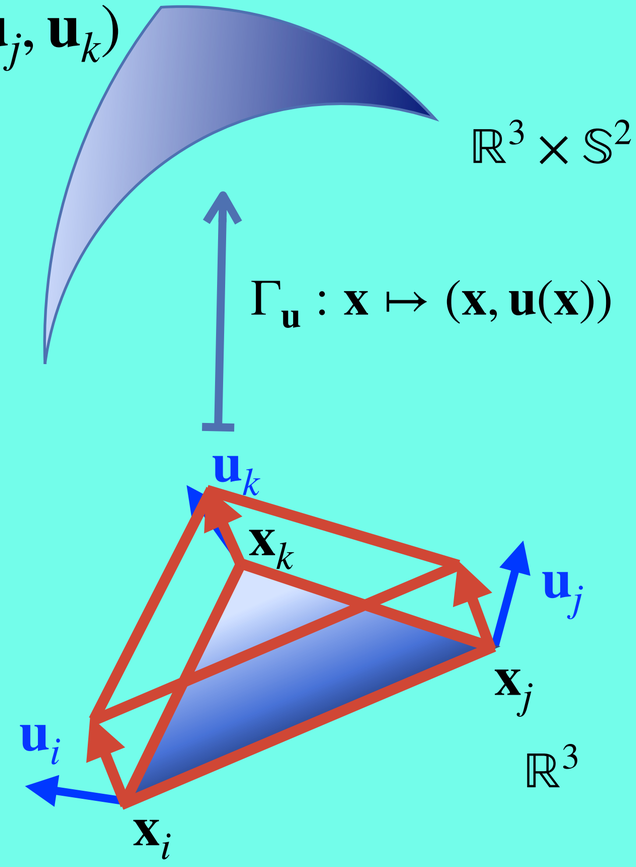
$$\mu^H_{\mathbf{u}}(\tau) = \int_\tau \Gamma^*_{\mathbf{u}} \omega^H = \frac{1}{2}\langle \bar{\mathbf{u}} \mid (\mathbf{u}_k - \mathbf{u}_j) \times \mathbf{x}_i + (\mathbf{u}_i - \mathbf{u}_k) \times \mathbf{x}_j + (\mathbf{u}_j - \mathbf{u}_i) \times \mathbf{x}_k \rangle$$

- *Gaussian curvature measure:*

$$\mu^G_{\mathbf{u}}(\tau) = \int_\tau \Gamma^*_{\mathbf{u}} \omega^G = \frac{1}{2}\langle \bar{\mathbf{u}} \mid (\mathbf{u}_j - \mathbf{u}_i) \times (\mathbf{u}_k - \mathbf{u}_i) \rangle$$

- *Anisotropic curvature tensor measure:*

$$\mu^{(\mathbf{X},\mathbf{Y})}_{\mathbf{u}}(\tau) = \frac{1}{2}\Big\langle \bar{\mathbf{u}} \mid \langle \mathbf{Y} \mid \mathbf{u}_k - \mathbf{u}_i \rangle \mathbf{X} \times (\mathbf{x}_j - \mathbf{x}_i) - \langle \mathbf{Y} \mid \mathbf{u}_j - \mathbf{u}_i \rangle \mathbf{X} \times (\mathbf{x}_k - \mathbf{x}_i) \Big\rangle$$



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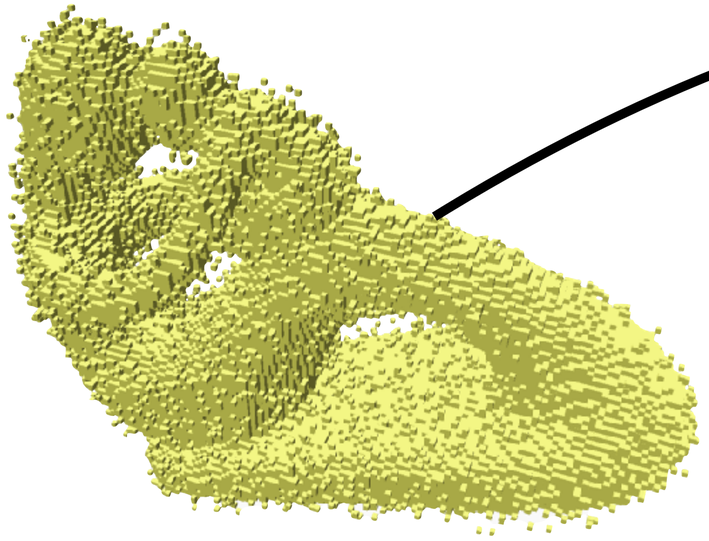
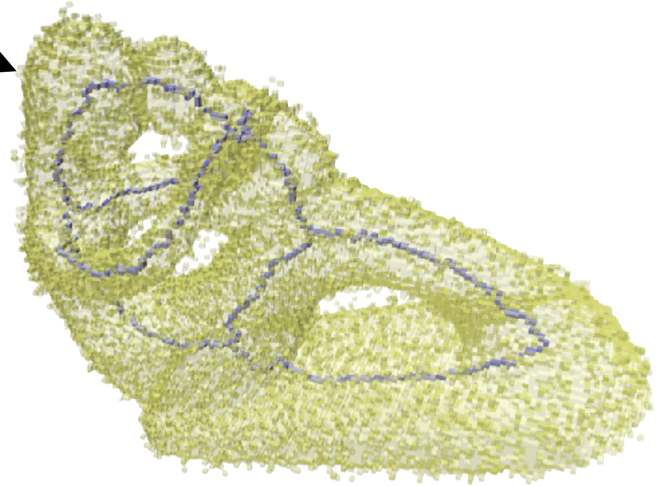
Link with persistent homology (topological data analysis)

Filtration: barcodes indicates birth/death of loops
Significant loops = longest loops

[Edelsbrunner, Letscher, Zomorodian 02]

Persistent homology





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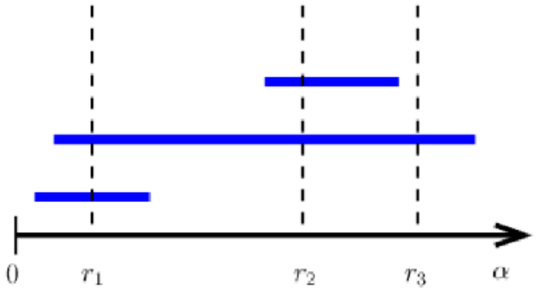

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