Group picture* with automata

 **

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La théorie des automates est la seule façon de parler de la pensée sans savoir ce qu'est "penser".

M-P. Schützenberger (?)

Plan

transducers

parity games

vector addition

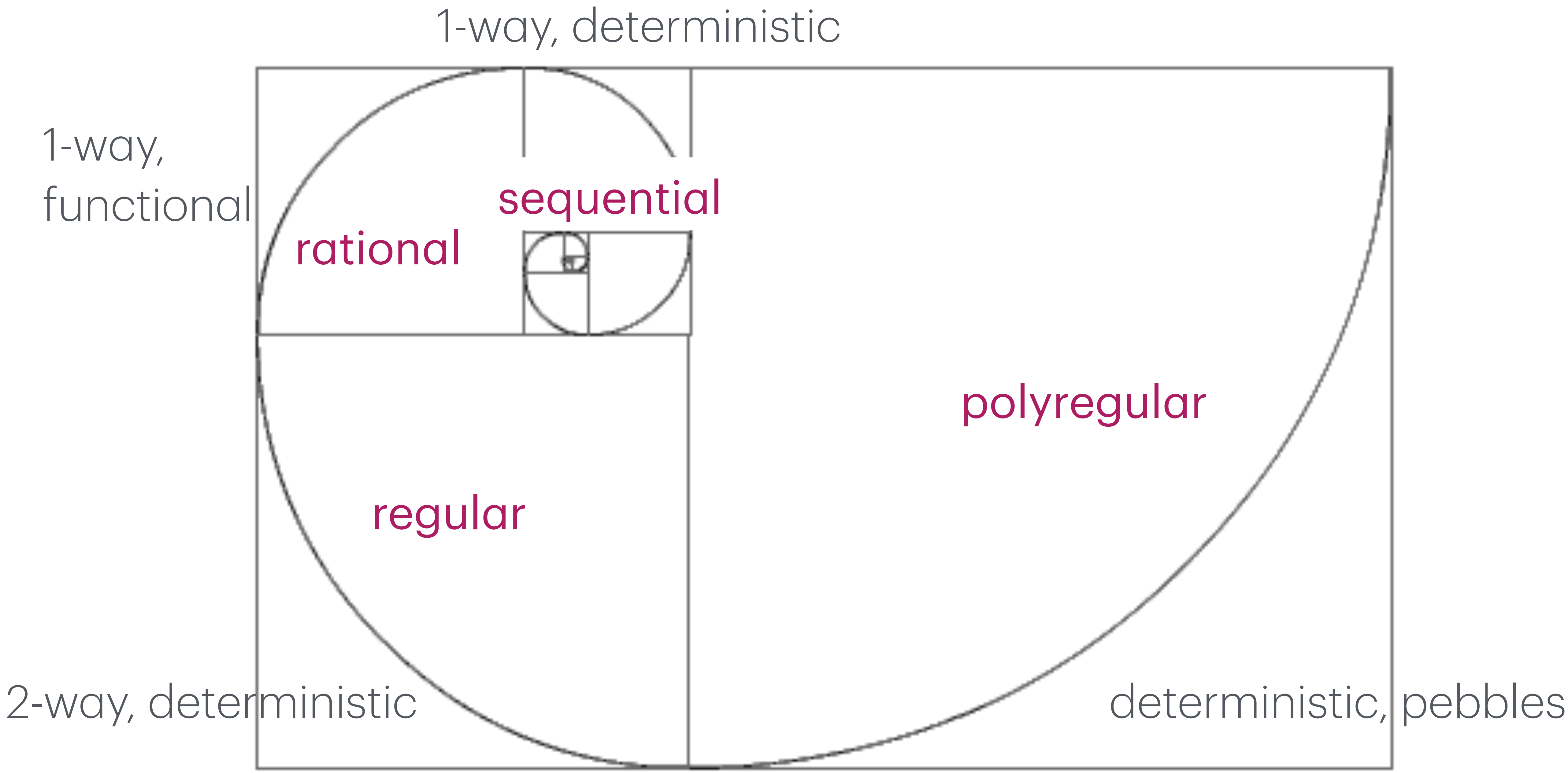
Transducers

Word transducers

- Moore (1956): Gedanken-experiments on sequential machines
- Schützenberger (1961): A remark on finite transducers
- Elgot-Mezei (1965): On relations defined by generalised finite automata
- Scott (1967): Some definitional suggestions for automata theory
- Nivat (1968): Transductions des langages de Chomsky
- Choffrut (1977): Caractérisation des fonctions (sous)séquentielles

Transducers

automata with outputs



Transducers

automata with outputs

1-way, deterministic

sequential

$a\,w \mapsto w\,a$

1-way,
functional

rational

$w\,a \mapsto a\,w$

poly-regular

deterministic, for-loops

$w \mapsto (w\,a)^{|w|}$

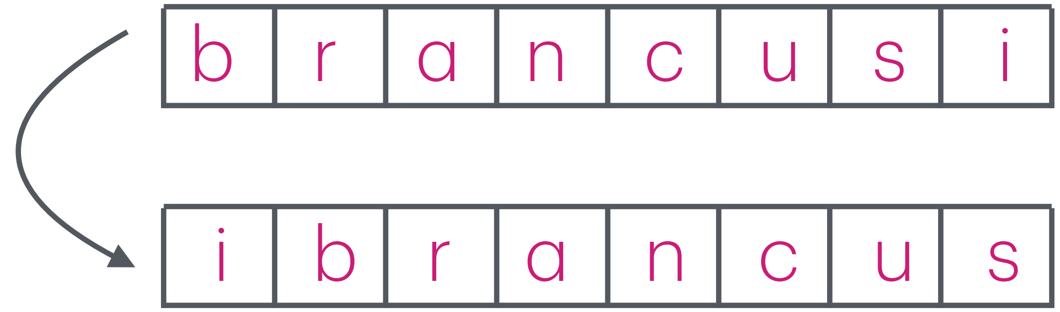
regular

2-way, deterministic

$w \mapsto \mathsf{mirror}(w)$

Transducers

automata with outputs



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deterministic, for-loops

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2-way, deterministic

$w \mapsto \mathsf{mirror}(w)$

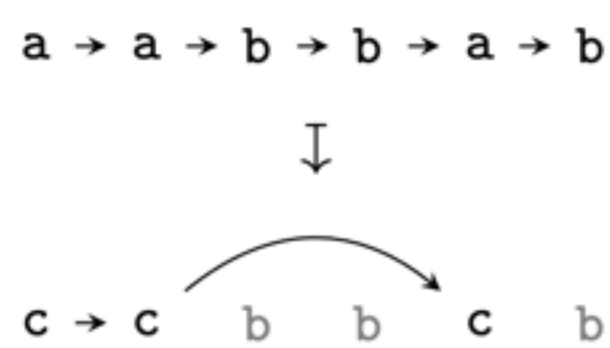
Regular transductions

Engelfriet-Hoogeboom (2001): **two-way and monadic second-order** transductions are equivalent.

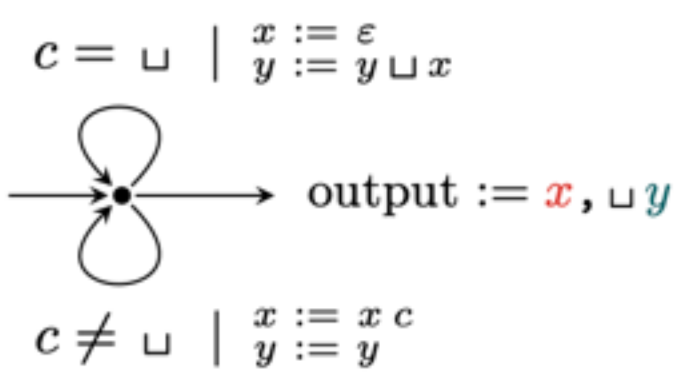
$$\varphi_{\text{pos}}(x) := a(x)$$

$$\varphi_{\text{c-lab}}(x) := \texttt{true}$$

$$\varphi_{\text{succ}}(x,y) := x < y \wedge a(x) \wedge a(x) \wedge \forall z \; x < z < y \rightarrow b(z)$$



Alur-Černy (2010): one-way transducers with **registers**



Alan␣M.␣Turing ↦ Turing,␣Alan␣M.

Polyregular transductions

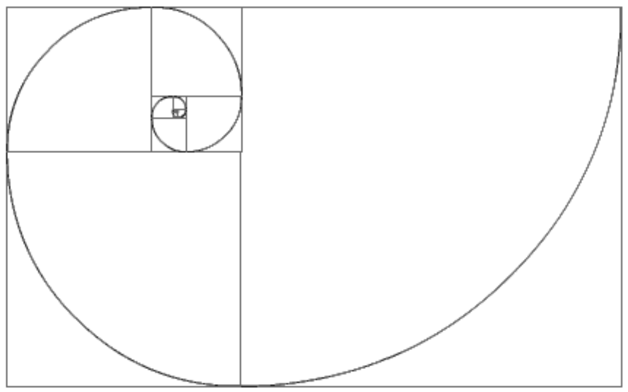
Bojańczyk (2018): transductions with **polynomial growth**

$$w \mapsto \underbrace{w \cdots w}_{|w|}$$

Equivalent characterisations (Bojańczyk 2018+Kiefer-Lhote):

- Nested pebbles
- **MSO interpretations** dimension corresponds to growth rate
- **For**-transducers

Results and questions



- **Class membership** (decidable)

  word combinatorics, degree MSO interpretations

- Closure under **composition** (holds)
  Reversible transducers: Dartois-Fournier-Jecker-Lhote 2017

- **Equivalence:** decidable on deterministic models, up to finitely-valued
  (Filiot-Jecker-Löding-M.-Puppis-Winter 2024)

Open: equivalence of polyregular transductions
(unary outputs, Douéneau-Tabot 2021)

Open: definability of regular transductions in first-order logic
(rational, Lhote 2018)

Open: canonical models for classes beyond rational transductions

Plan

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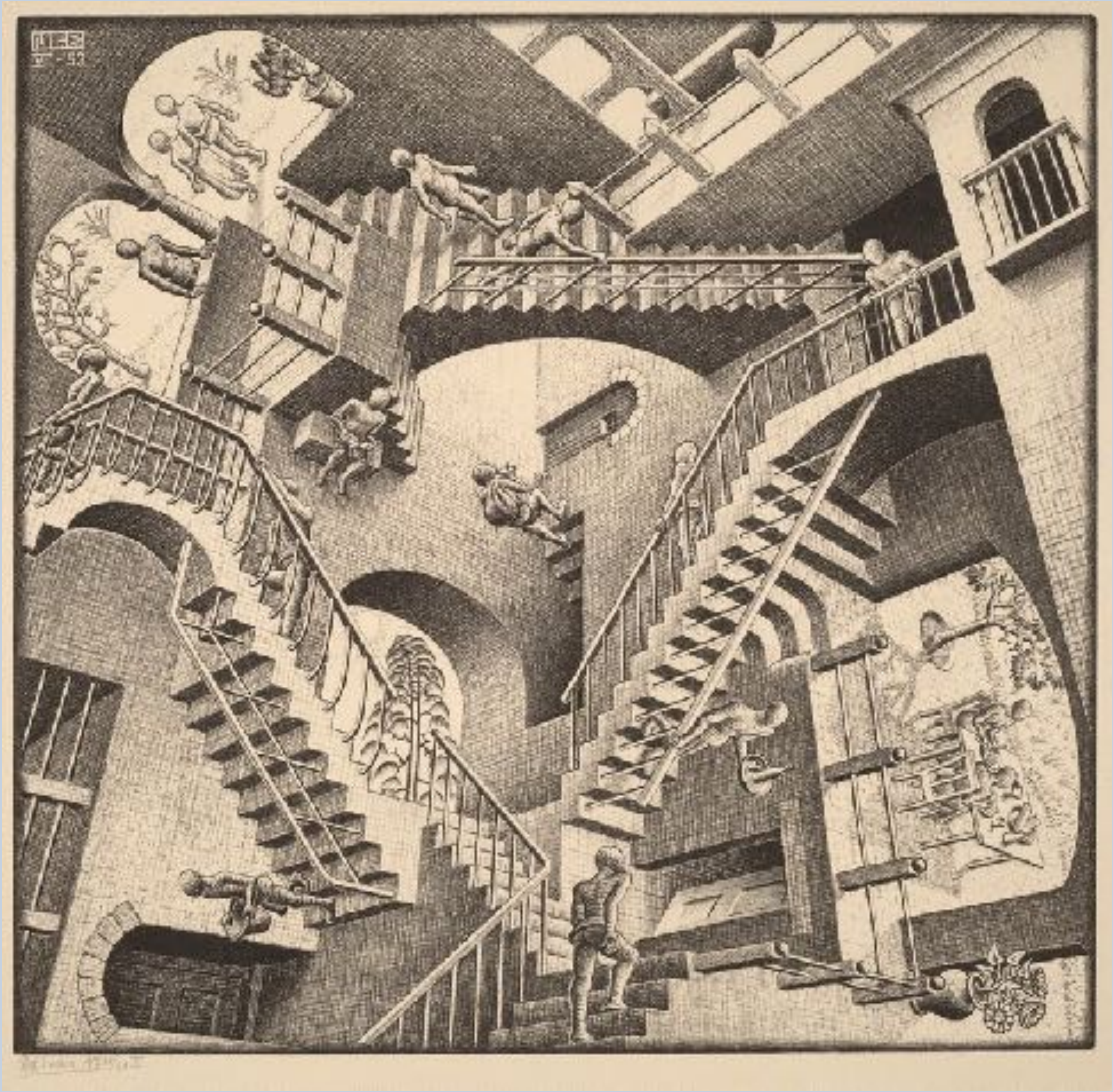
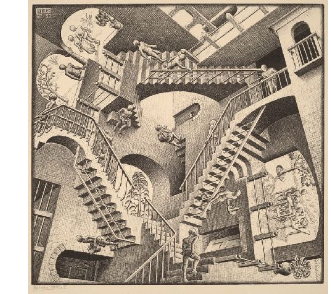
parity games

vector addition systems

Vector addition systems

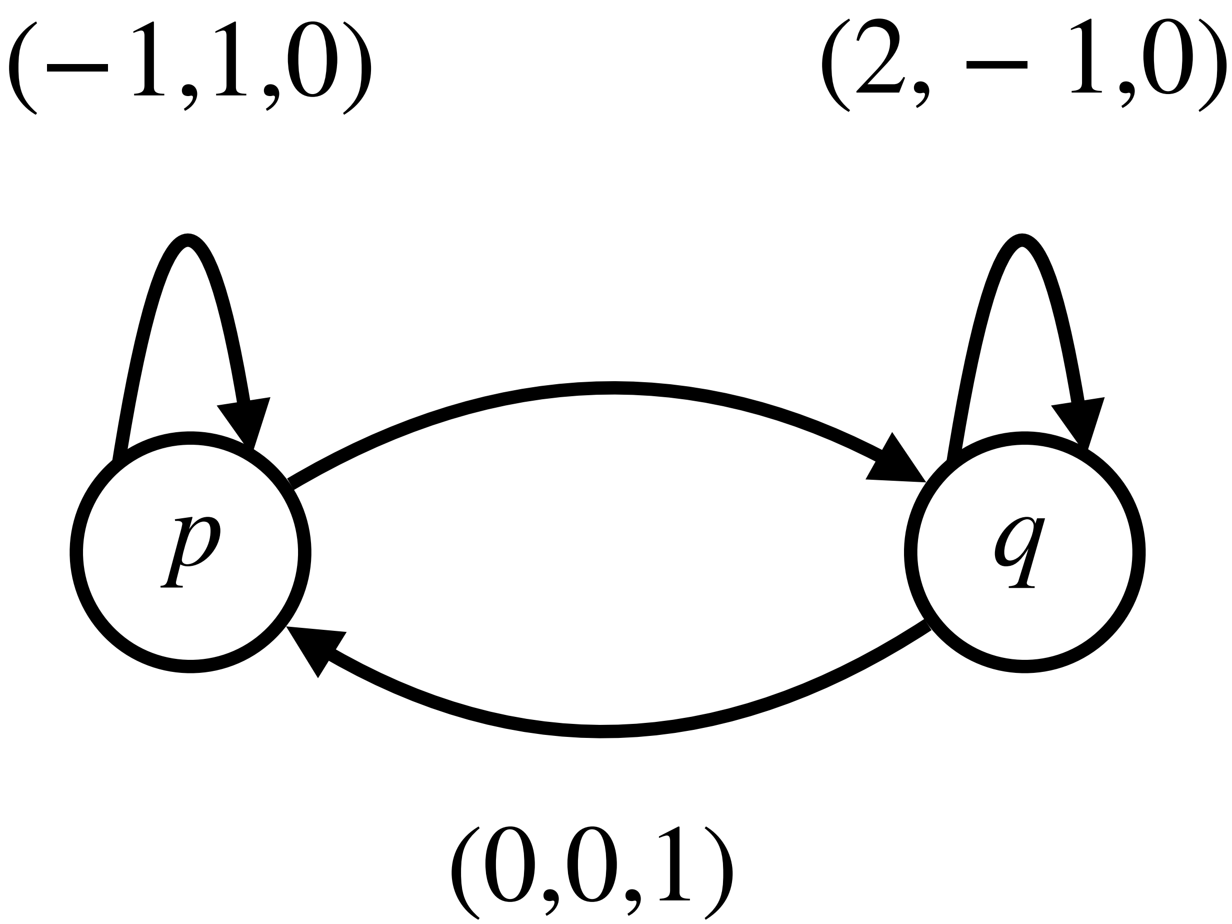


Vector addition systems



Equivalent: Petri nets, Minsky machines **without zero tests**

R



R*

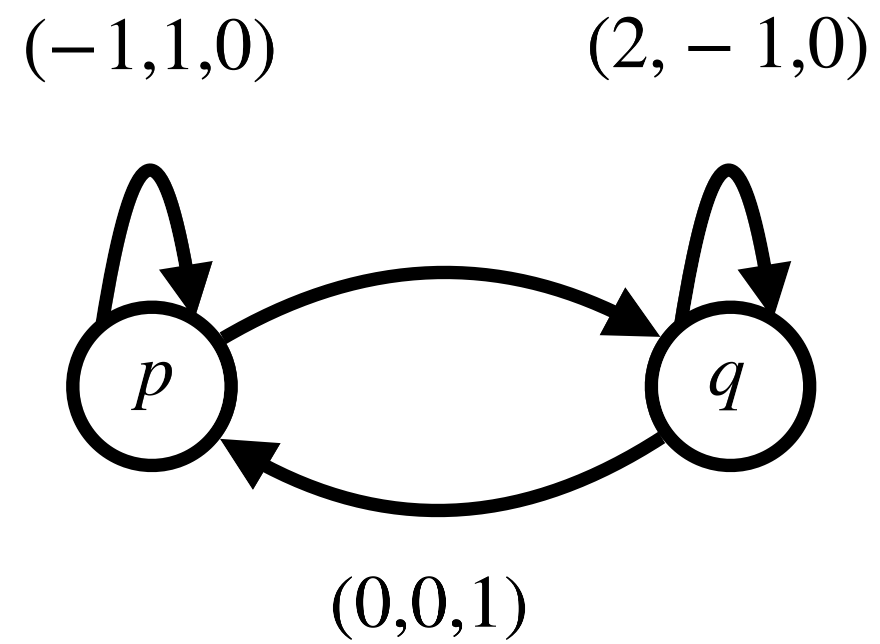
p(1,0,0)
↓
q(0,1,0)
↓
p(2,0,1)
↓
q(0,2,1)
↓
p(4,0,2)

R*(p(1,0,0)): p(x,y,z) with $1 \leq x + y \leq 2^z$

VASS reachability problem

Input: VASS **R** and two configurations **m, m'**

Output: yes if **R*(m,m')**



- ExpSpace lower bound (Lipton 1976)

- Decidable in dimension 2 (Hopcroft-Pansiot 1979, reachability set is effectively semi-linear).

- Decidability for arbitrary dimension: Mayr 1981, Kosaraju 1982

- Kosaraju-Lambert-Mayr-Sacerdote-Tenney approach till 2009

- New algorithm based on semi-linear invariants (Leroux 2009, 2011)

VASS reachability

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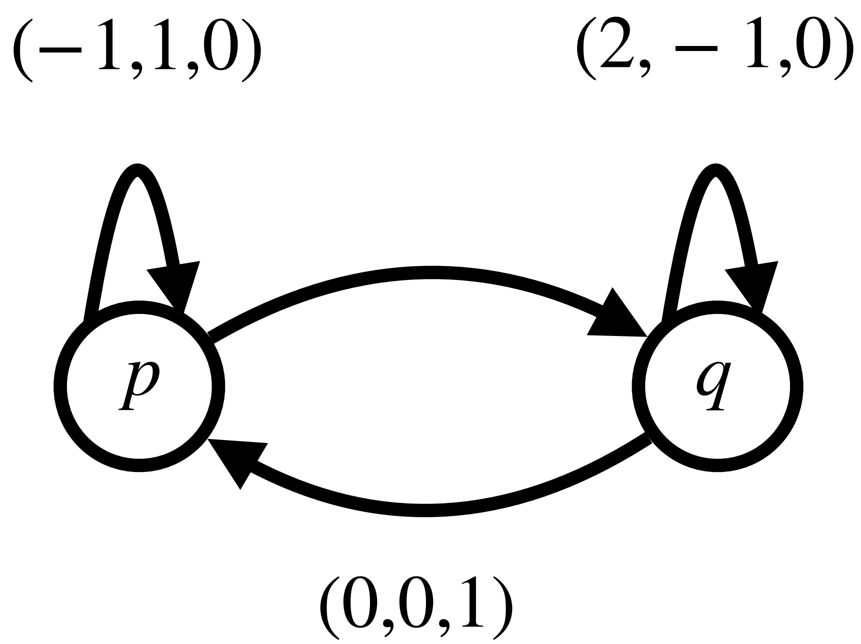


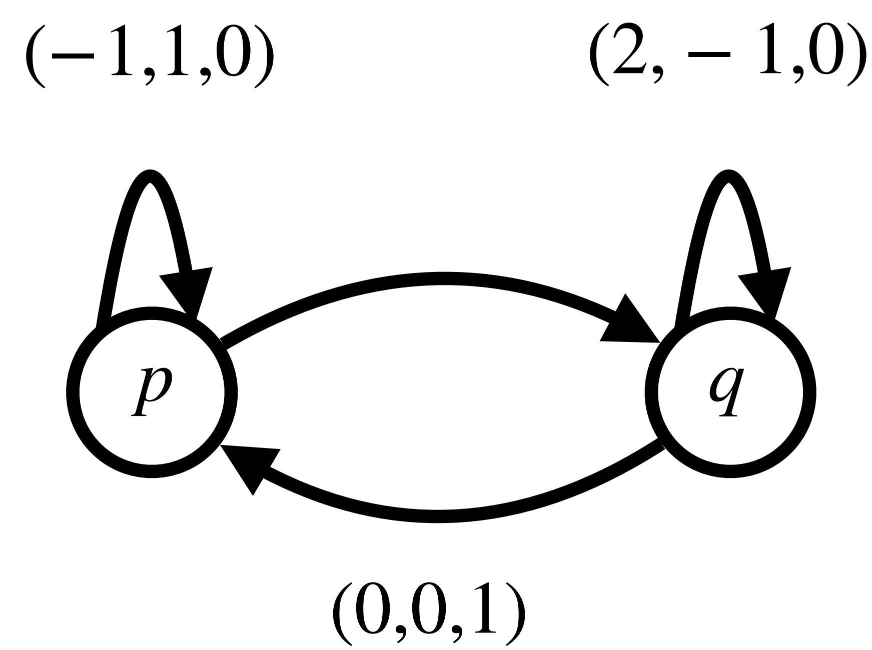
- Ackermannian upper bound: Leroux-Schmitz 2019

- Tower lower bound: Czerwinski-Lasota-Lazic-Leroux-Mazowiecki 2019

STOC best paper award

- Ackermann lower bound: Czerwinski-Orlikowski / Leroux 2021

VASS reachability



p(x,y,z) with 0 < x+y < exp(z) + 1

Input: VASS **R** and two configurations **m, m'**

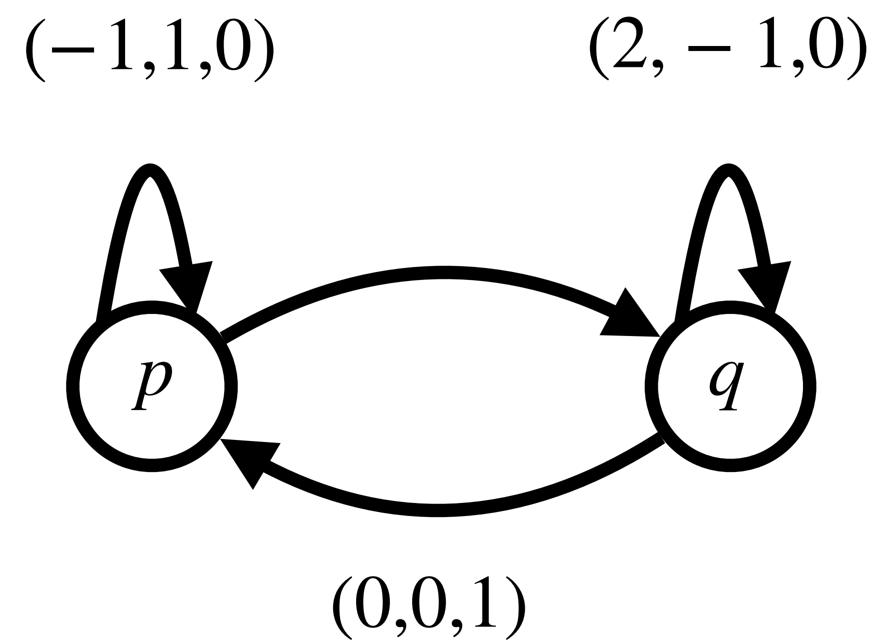
Output: yes if **R*(m,m')**

VASS Reachability (Leroux 2011):

Reachability sets are not semi-linear, but **almost semi-linear**

almost semi-linear = finite union of sets **b** + **P**, with **b**, **P** over $\mathbb{Z}^d$ and **P** is cone with $\mathbb{Q}_{\geq 0}$**P** definable in FO($\mathbb{Q}_{\geq 0}, +, 0, 1, \leq$)

VASS reachability



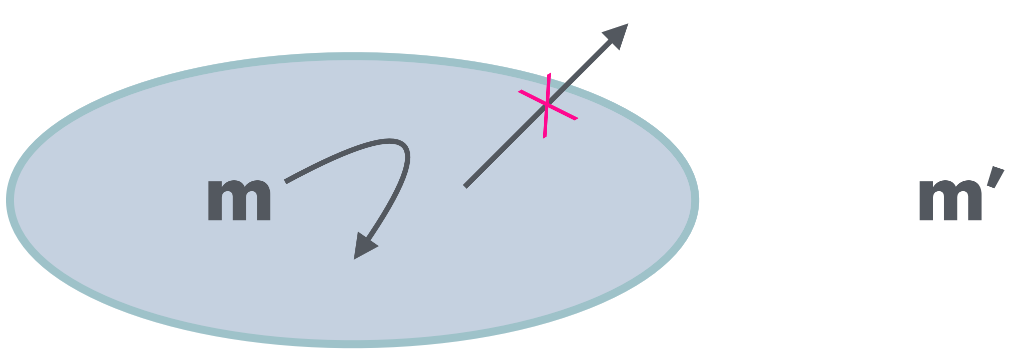
Input: VASS **R** and two configurations **m, m'**

Output: yes if **R*(m,m')**

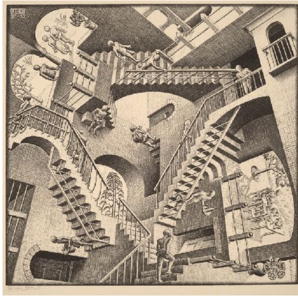
Simple algorithm (Leroux 2011):

p(x,y,z) with 0 < x+y < exp(z) + 1

Non-reachability can be witnessed by semi-linear inductive invariant:



2 simple semi-algorithms solving VASS reachability



VASS: perspectives

Extensions and problems:

- **Priority VASS**: counters are ordered. Counter k can be zero-tested if counters <k are all zero. (Reinhardt 1994, Bonnet 2011, Guttenberg 2024)

- **Pushdown VASS**: Reachability problem recently announced as decidable (2026).

- **Branching VASS**: runs are trees, counter values are added bottom-up. Reachability problem equivalent to provability in MELL (de Groote-Guillaume-Salvati 2004). Reachability problem is decidable for dimension 2 (Bizière-Leroux-Sutre 2025).

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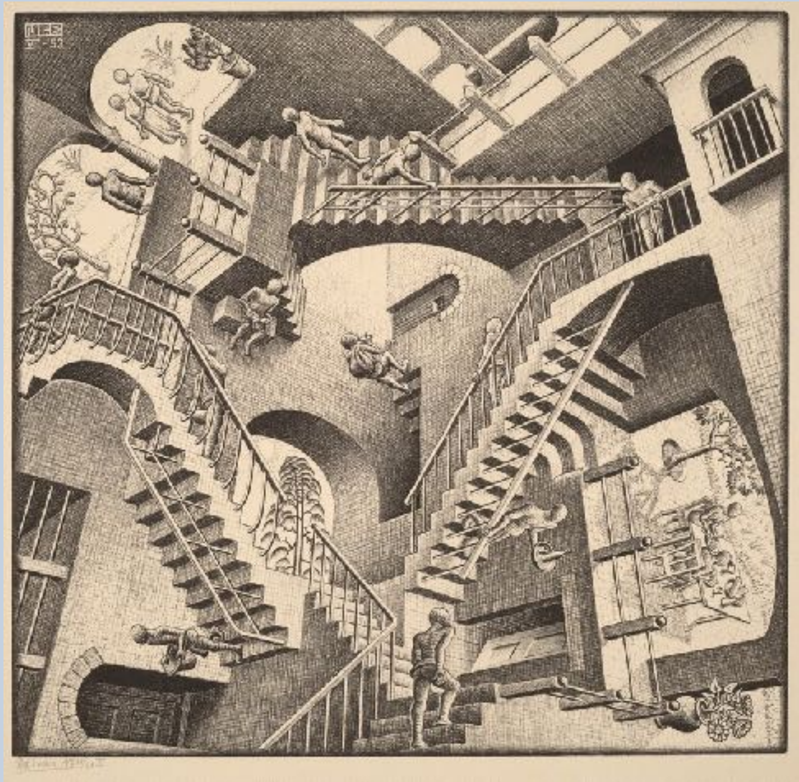


Games

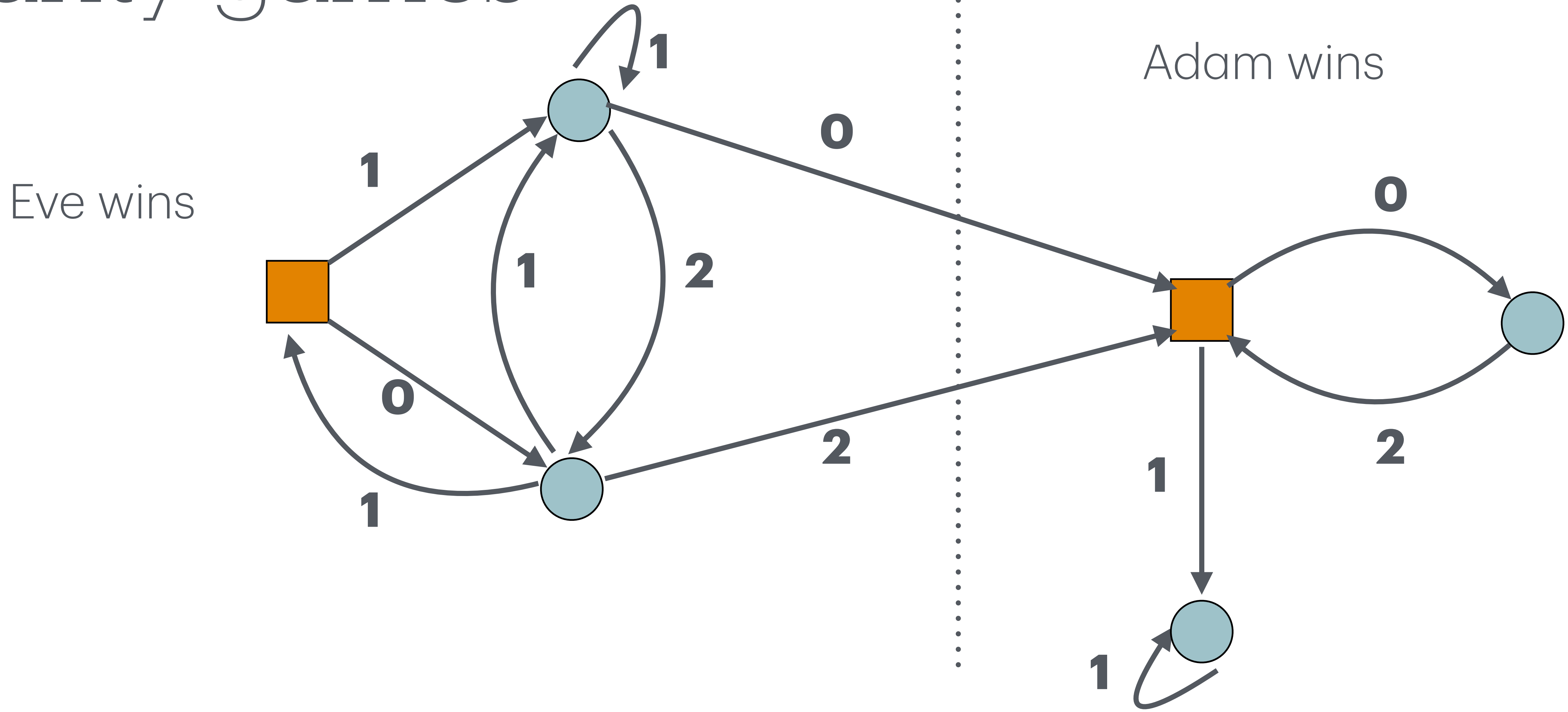
Logic

Automata

Logic, automata and games

- Church (1957): introduces the **synthesis problem** (build programs that are correct by construction)
- Büchi, Elgot, Trakhtenbrot (1958): equivalence of automata and monadic second order logic (MSOL) over finite words
- Büchi (1962): extends automata to infinite words
- Rabin (1969): extension to infinite trees, solution to synthesis problem using MSOL over trees
- Büchi, Landweber (1969): solution to synthesis problem using automata and **parity games**

Parity games



2 antagonistic players: Eve and Adam

Edges labelled by priorities

Eve wins a play if the maximal priority seen infinitely often is **even**

Parity games: algorithms

Input: game arena (|V|=n) with edges labelled by priorities 0 ... d

Output: vertices from which Eve wins

Parity games equivalent to mu-calculus model-checking

Parity games in NP and co-NP

McNaughton-Zielonka recursive algorithm (1998)

Jurdziński progress measure algorithm (2000)

poly(n), exp(**d**)

Calude et al. (2017), Jurdziński-Lazic (2017), Fearnley et al. (2017), Lehtinen (2018), Parys (2019)

$n^{\log(d)}$

Parity games: algorithms

Calude et al. (2017), Jurdziński-Lazic (2017), Fearnley et al. (2017), Lehtinen (2018), Parys (2019) $n^{\log(d)}$

All the above algorithms share so-called universal trees as combinatorial structure. There is a quasi-polynomial lower bound on universal trees. This explains why such algorithms cannot do better.

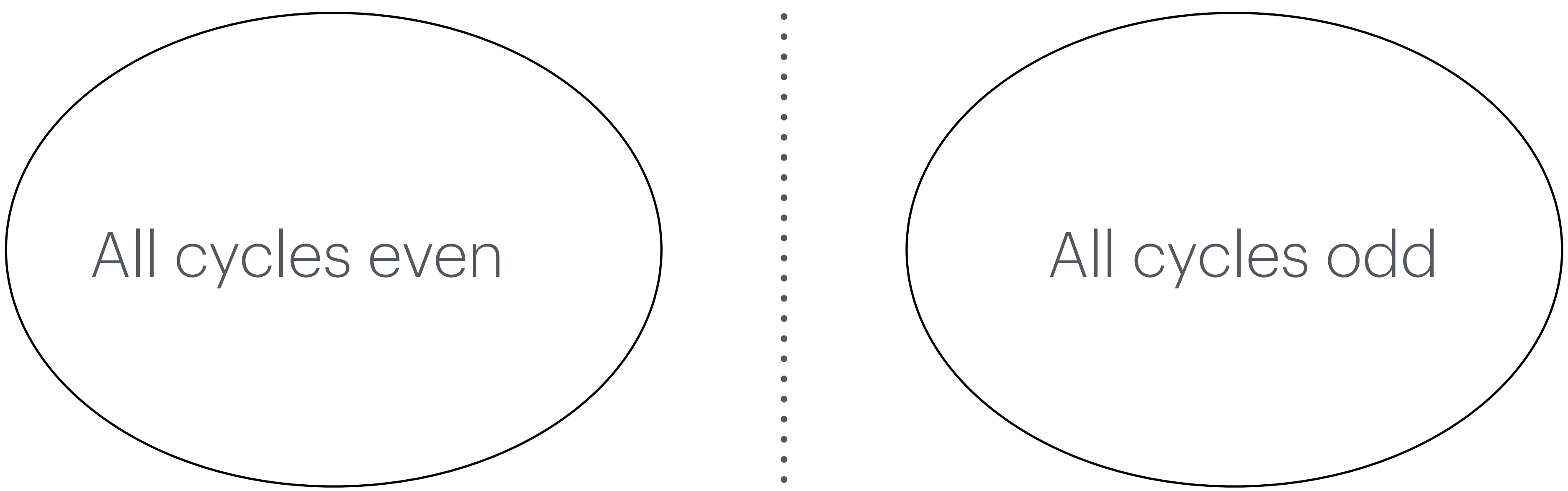
Czerwiński-Daviaud-Fijalkow-Jurdziński-Lazić-Parys (2019)
SODA best paper award

Parity games: universal trees

Bojańczyk-Czerwiński: reduction from parity to safety games via **automata**

PTime

Separating automata

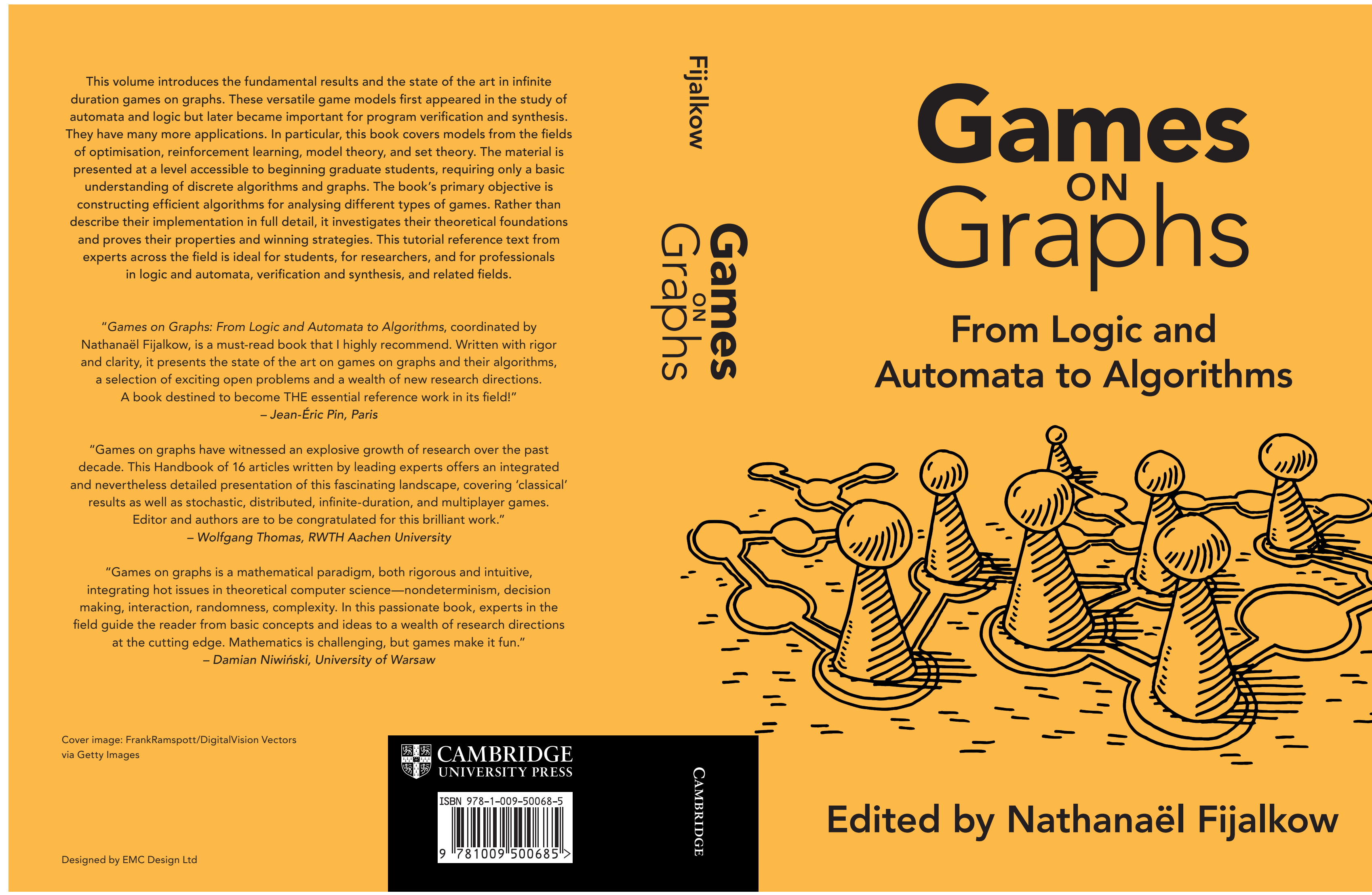


Every separating automata hides some universal tree. There is a quasi-poly lower bound on the size of universal trees.

embeds every tree

Games: Perspectives

Another 20 years for Parity Games?



Several other directions:

- narrowing down the complexity (PPAD, linear programming,...)
- memory for strategies (Ohlmann, Casares, Colcombet, Fijalkow, Bouyer, ...)

Back to finite automata: history-determinism

Automata over infinite words: canonical objects?

Parity automata: one-player games (Eve).

Deterministic parity automata = omega-regular languages.

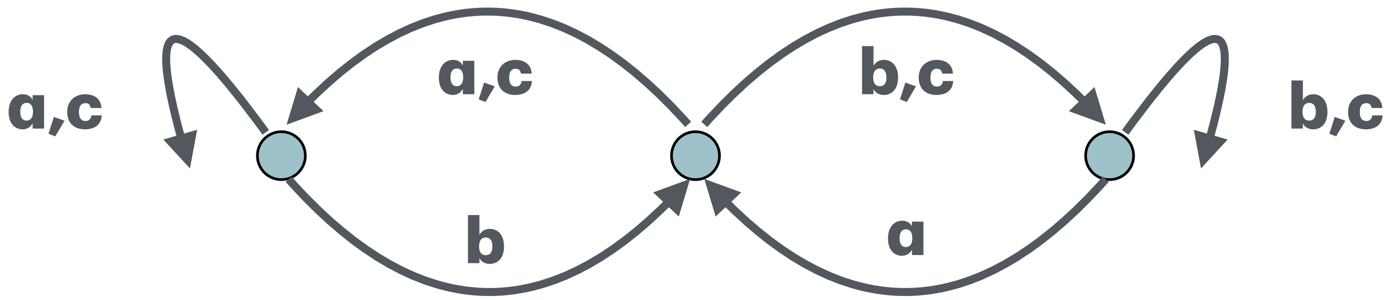
Deterministic parity automata: not canonical.

A non-deterministic automaton is history-deterministic if if there is some strategy to produce an accepting run, whenever there exists one.

history-determinism = good-for-games

(Henzinger-Piterman, Colcombet 2006/09)

History-determinism (HD)



finitely many **a**
or finitely many **b**

history-deterministic

Strategy in middle state for **c**: switch between **a,b**

Co-Büchi automata: HD checkable in PTime (Kuperberg-Skrzypczak 2015)

HD co-Büchi automata: PTIME minimisation (Radi-Kupferman 2022)

2-token game characterises HD (Prakash 2025, Bagnol-Kuperberg 2019)

Layered automata (Casares-Löding-Walukiewicz 2026): canonicity?

Conclusion

Personal selection of topics. Many more results around:

- weighted automata
- cost automata
- probabilistic automata
- partially observable games
- distributed games
- data logics and automata
- hierarchies
- algebra
- automata learning
- fine-grained complexity

Conclusion

You should always aim for the moon - even if you miss, you'll land among the stars. (O. Wilde)

Research is also about day-dreaming. It increases the chance to reach some stars along the way.