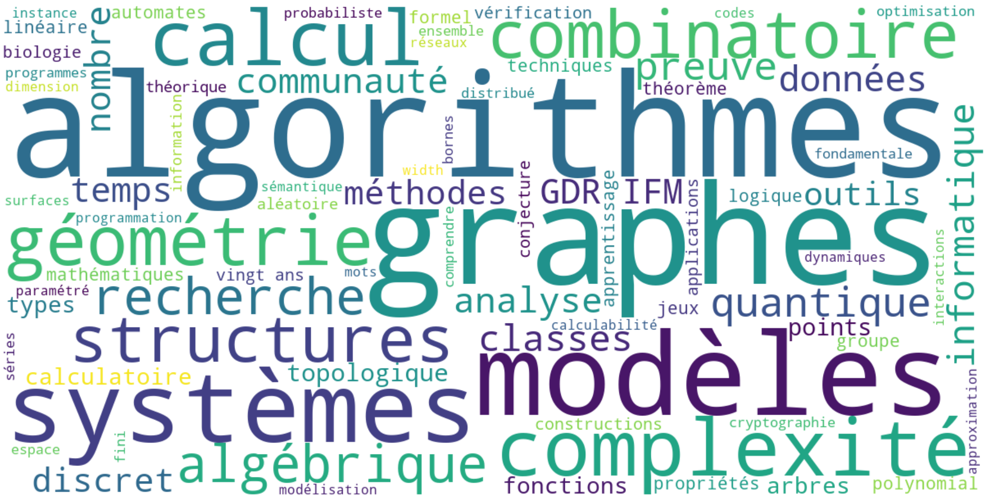
Groupement de recherche
IFM Informatique Fondamentale et ses Mathématiques

Les vingt ans du GdR IFM



Géométrie(s) et image

GT GéoAlgo (Géométrie algorithmique) 3
GT GDMM (Géométrie discrète et morphologie mathématique) 13
GT MG (Modélisation géométrique) 25

Twenty Years of GdR IFM, seen from GT Computational Geometry

General Presentation

The field of research of the GT GéoAlgo is *Discrete and Computational Geometry*.

Computational Geometry studies computational questions related to geometric objects. These questions often come from applied fields such as robotics (motion planning), computer graphics (mesh processing), data mining (multidimensional search), or optimization (linear programming). The focus is on provably correct algorithms with worst-case theoretical guarantees. Some algorithms are numerical and imply precision issues, but in most cases one assumes exact computations on real numbers, and the algorithms have a discrete flavor. *Discrete Geometry*, also called *Combinatorial Geometry*, studies combinatorial aspects of geometry. It is naturally very interrelated with Computational Geometry, and should not be confused with *Digital Geometry*, which studies discretization of geometric objects on grid-like spaces (with pixels and voxels). In the 2000s, *Computational Topology* emerged as a subfield of Computational Geometry and has grown significantly, in particular in the French community.

The flagship venue of the community is the *International Symposium on Computational Geometry* (SoCG). Additionally, publications occur in broader conferences in theoretical computer science and algorithms (STOC, FOCS, SODA), in machine learning (AISTATS, ICML, NeurIPS), and in more applied fields (Siggraph, SGP). Members participate in workshops in computer science (Dagstuhl) and mathematics (Oberwolfach). The community has dedicated journals (*Discrete & Computational Geometry* and *Journal of Computational Geometry*), but also publishes in journals with a broader scope (J. ACM, SIAM J. Comput., FOCM, and journals in algorithms, combinatorics, or geometry). The *Handbook of Discrete and Computational Geometry* [7] provides an excellent presentation of the field.

The French community plays a vibrant role on the international scene. It gathers researchers from CNRS (sections 2 and 3, and to a lesser extent section 1), Inria, and many universities, and holds a biyearly meeting, the *Journées de Géométrie Algorithmique*, supported by the GDR. Below, we review the main trends of the last 20 years, with a strong focus on the topics that are studied by the French researchers, and also emphasize connections with other working groups (GTs) of the GDR-IFM.

1 Geometric algorithms in $\mathbb{R}^d$ with theoretical guarantees

Delaunay triangulations, Voronoï diagrams, and polytopes.

These fundamental concepts are studied since the beginnings of Computational Geometry and they remain a vibrant area of research. Over the last twenty years, while new algorithms have been developed in the traditional Euclidean setting, the focus on Delaunay triangulations and Voronoï diagrams has also widely expanded: refined algorithms have been developed for weighted and constrained variants (see e.g. [11] for an application in computational optimal transport), for robustness in practical meshing pipelines, and for extensions to more complex metric spaces—including curved geometries such as hyperbolic surfaces (see below). At the same time, the study of convex polytopes has continued to grow, driven by the need to handle high-dimensional data and models. Because polytopes admit different representations whose sizes can significantly vary, designing efficient algorithms to approximate a polytope—or even estimate its volume—has become a key challenge. These lines of work illustrate how classical geometric structures continue to evolve in response to modern computational demands.

Geometric algorithms.

Standard algorithmic problems are revisited in a geometric context. Prominently, any family of geometric objects can be studied through its *intersection graph*, and classical packing and covering problems (and variations) are attacked, sometimes in an approximate way, by exploiting properties of the graph itself and by using other properties and concepts (VC-dimension, ε-nets, ...). For this purpose, algorithms on intersection graphs, e.g. for coloring, TSP, and separators, have been developed. As a particularly fruitful example, efficient approximation schemes have been discovered recently using local search, leading to algorithms that are very simple (but highly non-trivial to analyze). For the specific case of the plane, other types of graphs are relevant, such as *visibility graphs*, whose construction and properties have been widely studied until around 2015.

Numerical and symbolic computations.

Most geometric algorithms are designed under the assumption that computations are performed over the real numbers, while practical implementation manipulate fixed-precision floating-point numbers, integers, or algebraic numbers. This discrepancy may yield undesirable behavior (crashing, entering infinite loop, producing incorrect output) and addressing this properly led to the development of the *exact geometric computation* paradigm at the core of the CGAL library. Evaluation of geometric predicates (e.g., deciding whether a point does belong to a line defined by two other points) is not spared, yielding important interactions between the GTs “Calcul Formel” and GéoAlgo. A notable example was identified in 2018: a vulnerability in Skia graphics library (used by Chrome etc.) caused by a non-state-of-the-art implementation of a 2D convex-hull algorithm. Key results in the field include the first algorithm to remove intersections in 3D triangle meshes with fixed-precision output vertices, addressing a long-standing practical need for industrial and academic implementations. Another result is the development of efficient algorithms for drawing algebraic plane curves, approximating both the curves and self-intersections while preserving topology and controlling approximation error. A further achievement is the first practical and robust computation of 3D quadric intersections using algorithms that reduce the degree of algebraic numbers involved. A recent trend focuses on hyberbolic geometry where cascaded constructions of points make it critical to design algorithms with predicates and algebraic numbers of low degree.

Probabilistic analysis of geometric algorithms.

Over the past two decades, the probabilistic analysis of geometric algorithms has gained prominence as a way to temper the often pessimistic nature of worst-case bounds—such as the classical exponential upper bounds for the simplex method. This shift has motivated the introduction of more realistic input models and average-case analyses. A major conceptual breakthrough in the early 2000s was the development of smoothed analysis, which blends worst-case and average-case perspectives; its application to the simplex algorithm by Spielman and Teng had a profound impact on the field. In France, this line of research fostered closer interactions between communities (GTs) in analysis of algorithms and stochastic geometry, notably through several national collaborative projects in the 2010s. Over the last fifteen years, French researchers have made significant contributions in low-dimensional settings, including advances in smoothed analysis, probabilistic approximation of polytopes, and the study of random walks in geometric contexts. These works illustrate the increasingly rich interplay between geometry, randomness, and algorithmic efficiency.

2 Meshes, triangulations, geometric and topological inference

Meshes/triangulations and in particular *Delaunay complexes* have been a cornerstone of computational geometry from its inception. The interest in meshing is motivated by e.g. applications in numerical PDEs (finite element methods) and graphics. Up to the year 2000 the study of meshing was limited to low dimensions and mostly Euclidean space. Since then, the research has gone into new directions with an emphasis on high dimensions and curved spaces: surfaces, hyperbolic spaces, and more generally Riemannian manifolds. The manifolds we want to mesh occur in various forms: submanifolds that are known from samples (this combines meshing with shape reconstruction discussed below), submanifolds that are given by (not necessarily algebraic) equations, or abstract Riemannian manifolds (given by local charts ignoring the ambient space, which avoids the curse of dimensionality). For instance, one important line of work has been to adapt the classical algorithmic toolbox for Delaunay triangulation so that it works beyond Euclidean settings. This has been initiated in toroidal and hyperbolic geometries, both of which are relevant for applications, e.g., in materials sciences, and many questions still abound in this active area of research. The works in the community have many facets: algorithm development, theoretical guarantees of correctness, complexity theory, implementation and practical running time.

Shape reconstruction is a classical problem in computational geometry, closely related to meshing. Given a finite set of data points that sample an unknown shape, the problem consists in building an approximation (ideally a triangulation) of the shape based solely on the data points. In the early 2000s, the first methods with theoretical guarantees were proposed for surfaces in $\mathbb{R}^3$. Around 2005, the focus started to shift toward *manifolds* in high dimensional ambient spaces, motivated by applications to *machine learning*. A milestone was obtained with the work of Niyogi, Smale and Weinberger in 2008, which provided sampling conditions under which an offset of the data points recovers the homotopy type of the underlying manifold [16]. Since then, significant effort of the French community has gone into generalizing and improving the bounds of that seminal paper. In a high-dimensional ambient space, classical algorithms based on the computation of the Delaunay complex of the data points suffer from the curse of dimensionality. Alternative methods have been proposed that scale well with the ambient dimension and come with guarantees [1]. This led to developing new data structures for encoding simplicial complexes: the simplex tree and the blocker-skeleton data structure, which are both at the core of the Gudhi library (cf. §3 and 6). In a recent breakthrough, manifold reconstruction has been expressed as finding minimal cycles (either for an ℓ_1-norm or for a lexicographic order, yielding a topologically correct and efficient algorithm for surfaces in $\mathbb{R}^3$. The geometric assumptions that underpins the correctness of many algorithms mentioned above are most often formulated in terms of the reach [6]: the largest distance under which a point is guaranteed to admit a unique projection on the manifold. A new metric characterization of the reach was provided in [2] which proved useful for (statistical) learning. The reach is intricately linked with the medial axis, a skeleton which lies in the middle of a shape and captures the homotopy type and plays an important role in graphics, though being difficult to compute. Significant effort has gone into computing and simplifying the medial axis.

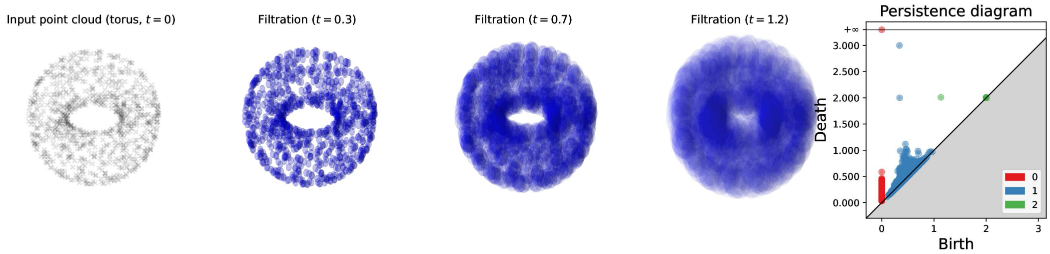


Figure 1 Čech filtration and resulting persistence diagram built on top of a point cloud (1000 points sampled uniformly on the surface of a torus). The right plot depicts the resulting PD: red points correspond to homology of dimension 0 (connected components), blue points to dimension 1 (loops) and green points to dimension 2 (cavities). The three points far away from the diagonal (the two blue points and the green one) testify for the two loops generating the torus and the corresponding cavity, points closer to the diagonal (birth $\simeq$ death) correspond to features that are merged quickly after appearing in the filtration (i.e. persisted less longer), often considered as topological noise.

3 Topological Data Analysis

Topological Data Analysis is a relatively recent machinery in computational topology that aims at computing *quantitative* topological descriptors of structured objects (graphs, points sampled on a manifold, time series, etc.). The field emerged in the early 2000s, and has since found application domains such as computer graphics, material science and computational biology to name a few. Rooted in algebraic topology, and more precisely in the theory of *persistent homology*, the TDA pipeline can be summarized in the following way: given a topological space $\mathcal{X}$ and a map $f : \mathcal{X} \to \mathbb{R}$, consider the *filtration* $K_t = \{x \in \mathcal{X},\ f(x) \leq t\}$. For instance, one may take $\mathcal{X} = \mathbb{R}^d$ and $f(x) = \min_{1 \leq i \leq n} \|x - x_i\|$ where $X = \{x_1, \ldots, x_n\} \subset \mathbb{R}^d$ is a point cloud, yielding the so-called Čech filtration. One then records the pairs of values (t_b, t_d) at which a given topological feature (such as connected components, loops, cavities, etc.) appears in K_{t_b} and disappears in K_{t_d}.[1] The resulting (multi-)set of pairs is called the *persistence diagram* (PD) $\mathrm{PD}(f)$ of f. See Figure 1. A fundamental property of PDs is their *stability*: similar filtrations yield similar PDs [3], where similarity of PDs is measured using optimal transport-type distances.

By the mid-2010s, PDs began to be used in machine learning as static features derived from data. Because the space of PDs has a non-linear structure, their integration into downstream models is non-trivial, and typically relies on vectorizations techniques, such as kernel methods, or data-driven representations based on specific deep neural network architectures. More recently, to further strengthen the connections between TDA and machine learning, the concept of persistence-based *topological optimization* has emerged. A rigorous framework for differentiating composite maps of the form $f \mapsto \mathrm{PD}(f) \mapsto \ell(\mathrm{PD}(f)) \in \mathbb{R}$ for some loss functions ℓ as been introduced in [15]. A series of works has since been devoted to studying, implementing, and refining gradient-based optimization schemes—an area that continues to be particularly active within the French TDA community.

In addition, the field has seen significant advances in the context of multiparameter

[1] In practice, K_t is represented by a combinatorial object called a simplicial complex (which should have, or closely approximate, the same homotopy type as the corresponding sublevel set), and PDs are computed using matrix reduction algorithms.

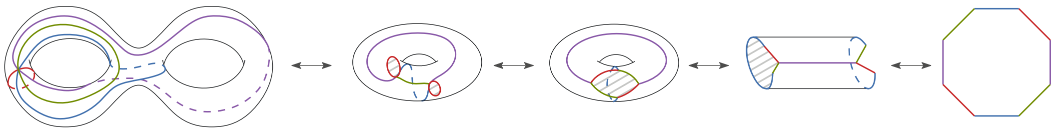


Figure 2 Cutting a double-torus (an orientable surface of genus two) into a disk using a *polygonal schema*.

persistence, that is, when the map f is valued in $\mathbb{R}^d$ with $d \geq 2$. In contrast to the case $d = 1$, the lack of total order on $\mathbb{R}^d$ prevents one from defining canonical analogues of PDs. The design of topological features based on multiparameter persistence has been studied deeply in the field, with the French community being a driving force, with the design, theoretical analysis, and implementation of two topological features for multiparameter persistence: the *signed barcode* and *candidate decompositions*. Multiparameter topological optimization has then quickly followed as a new application field in TDA, fueled by these new features.

These efforts materialize by the development of the open-source library Gudhi (`Python` / `C++`), which contains most of TDA descriptors with fast implementations in `C++`, as well as easy-to-use pipelines in `Python` for data science. As such, it has become the leading library for TDA, with thousands of downloads per week. With the same ambition, the development of multiparameter persistence has been made easier thanks to the multipers library, whose core implementations rely on `Gudhi`.

4 Low-dimensional computational topology

While algorithmic aspects are implicit in topology since its inception, a systematic development of algorithms for topological problems emerged only in the late 1990s [5]. The French community is particularly active in low-dimensional computational topology, where the ambient space is a surface or a 3-manifold (a space locally homeomorphic to $\mathbb{R}^3$), with some non-trivial topology (e.g., "surfaces with handles" such as the torus). Like topological data analysis, this field relies on algebraic topology, but it has a very distinct flavor: the applications are very different, and low dimension allows us to work with homotopy, which is finer than homology. The relevant tools from computer science include graph algorithms, complexity theory, and fixed-parameter tractability. In this area, the GT-GéoAlgo has connections with several other GTs, notably CoA and Graphs, and is also related to other fields, such as graph drawing. Relevant applications are found in computer graphics, geometry processing, and molecular biology.

The **2-dimensional case**, namely computational topology for graphs on surfaces [4], started to flourish 20 years ago [14]. The most basic questions are algorithmic formulations of well-studied mathematical problems: Are two input curves or graphs on a surface equivalent, for various topological notions of equivalence (homotopy, homology, isotopy)? Given a self-intersecting graph or multicurve, what is the minimum number of crossings that must remain after homotopically deforming it on a surface? Around the same time, algorithms for computing shortest curves and graphs with given topological properties blossomed, e.g., shortest non-contractible or non-separating closed curves; this requires an appropriate discretization of a metric surface.

While the above line of research is still active, several directions have emerged more recently. Applications to graph problems, e.g., generalized cut and flow problems, have been found: combining topological subroutines with tools from algorithmic graph theory (treewidth, minors, etc.), faster algorithms are obtained assuming the input graph is planar or, more generally, is embedded on a fixed surface—often these algorithms are fixed-parameter tractable in the genus of the surface. Generalizations of planar structures (e.g., Schnyder woods) to surfaces are investigated. Compact data structures for surface triangulations are discovered. Efficient algorithms are described to compute well-studied mathematical quantities related to surfaces, such as their spectra. Algorithms and data structures are being developed to handle hyperbolic surfaces and compute Delaunay triangulations and Voronoi diagrams on such spaces.

In the **3-dimensional case**, new topological features and knotting phenomena start to appear. Their investigation from a computational point of view was initiated about twenty years ago in a seminal work [10] and our understanding has been steadily improving since then, see for example [13]. In terms of exact recognition algorithms, this area is populated with peculiar questions at all ends of the complexity spectrum. For instance, the problem of deciding if a closed curve is knotted is now known to be in NP $\cap$ co-NP but no polynomial-time algorithm is known, while the best-known algorithm to check whether two knots are equivalent is elementary recursive but the problem is not even known to be NP-hard. Topological questions in 2D on homotopy, homology, etc. also adapt naturally to 3-manifolds, leading to new problems which are often harder to solve algorithmically. Over the past decades, the computational geometry community in France and abroad has largely contributed to mapping the 3-dimensional complexity landscape, in particular through the design of hardness proofs.Another computational aspect of this field is that topological insights are often garnered via the development of highly-optimized specific software tailored for the analysis of knots and 3-manifolds, which then fuels experimental works.

Given the dearth of knowledge on exact recognition algorithms, it is natural to investigate *invariants* which provide a coarser way to differentiate topological objects. Topological invariants of algebraic nature, such as quantum invariants of knots and 3-manifolds whose introduction dates back to the 80s, have been recently studied from the computational point of view. Their computational hardness has been established thanks to connections with quantum computing [12], and efficient algorithms have been designed through the extension of techniques from parameterized complexity. This phenomenon has, in turn, fueled a new interest for the study of the structural properties of combinatorial representations of knots, 3-manifolds, and other low dimensional objects, notably with the extension of techniques from structural graph theory to the context of low dimensional topology, or the study of the complexity of hard problems on inputs of restricted topology.

5 Discrete Geometry, geometric and topological combinatorics

Discrete geometry studies the structure of discrete objects in a geometric space, from finite point sets in the plane to arrangements of n-dimensional convex bodies. Foundational problems in this area deal with packing (Kepler's conjecture), polytopes and triangulations (Hilbert's third problem on decomposing polyhedra, Hirsch's conjecture on the diameter of polytopes, the g-conjecture on the face numbers of triangulated spheres), point configurations (the Erdös-Szekeres problem, distinct and repeated distances, integer distance sets), etc. There are strong ties between discrete geometry and computational geometry around questions of complexity, simulation, discretization, etc. and the two communities merged to some extent in the 1980's.

Within the French community, several lines of research in discrete geometry have been particularly active over the last two decades. Disk and ball packings in dimensions two and three continue to attract attention: although the monumental proof of the Kepler conjecture [9] settles the case of a single radius, the landscape becomes far richer when balls with multiple radii are allowed, raising challenging questions of optimal structure and computational optimization. Progress has also been made on the study of oriented matroids, a combinatorial and algebraic structure arising from point configurations and hyperplane arrangements, with connections to the complexity class ETR (Existential Theory of the Reals) [17], to probabilistic properties of geometric structures, and to refined combinatorial classifications. Hitting sets problems (finding a small number of points/lines/hyperplanes that intersect all elements of a collection of sets) have likewise seen strong advances, especially regarding the existence and computation of hitting sets and the influence that the geometry (convexity, half-spaces, etc.) has on the size of hitting sets. Equipartition problems (seeking for a single hyperplane cutting a collection of objects in halves, e.g. ham-sandwich theorem)—often involving partitions by systems of hyperplanes—have developed new links to classical conjectures such as Mahler's. Beyond these themes, new combinatorial objects inspired by geometric constructions have been introduced, including generalized associahedra and related polyhedral families, and have deepened the study of reconfiguration graphs such as flip graphs of triangulations and decomposition complexes, often in connection with low-dimensional topology (Section 4).

In the last 20 years, the development of algebraic and topological methods have been two driving forces in discrete geometry. A spectacular example was the *polynomial method* [8] which builds on ruled surface theory in real algebraic geometry and led to solutions to the *joint problem* and Erdös' *distinct distances problem* and the algorithmic technique of *polynomial partitioning*. Another landmark is Adiprasito's proof of the *g-conjecture*, which draws on the connections between face rings and toric varieties and yields extensions to simplicial complexes of some classical results on graphs (*e.g.* a linear bound on the number of facets in terms of the number of ridges, a generalization of Heawood's inequality on which complexes embed in manifolds, etc.). Other examples include the sharpening of several fundamental results in extremal combinatorics for *semi-algebraic (hyper)graphs*, the developments around the *configuration space/test map* scheme (*e.g.* the topological Tverberg theorem), etc.

6 Software development in Computational Geometry

The French community is involved in the development of several open-source software related to computational geometry: gudhi for Topological Data Analysis, contributions to `Maple` (root finding algorithms), Isotop (drawing of planar curves with correct topology), to name a few.

A major contribution of the French community to discrete and computational Geometry over the past decades lies in the development and long-term maintenance of the Computational Geometry Algorithms Library (CGAL). Initiated in 1996, CGAL has grown into the most influential and widely used software library in computational geometry. It provides robust, efficient, and well-documented implementations of geometric algorithms for both academic and industrial use. The project now comprises more than 700,000 lines of code, is downloaded about 10,000 times per year, and serves over 200 commercial users worldwide. With around 20 active developers and supervised by an editorial board in charge of project management and code review, CGAL continues to be a vibrant and sustainable initiative, supported by academic partners who have made long-term commitments to its maintenance and evolution. The library is distributed under both open-source and commercial licenses, ensuring accessibility for research as well as reliability for industrial applications. The private company GeometryFactory, an integral partner of the project (including 3 members of the editorial board), provides professional support and assists industrial users in integrating CGAL components into their software. In recognition of its enduring impact, CGAL received the "Test of Time Award" at the 2023 Symposium on Computational Geometry (SoCG), underscoring its role as a cornerstone in the field's computational infrastructure.

Contributors.

Dominique Attali, Mathieu Carrière, Xavier Goaoc, Théo Lacombe, Clément Maria, Arnaud de Mesmay, Guillaume Moroz, Mathijs Wintraecken.

References

1. Jean-Daniel Boissonnat, Frédéric Chazal, and Mariette Yvinec. *Geometric and Topological Inference*. Cambridge University Press, 2018.
2. Jean-Daniel Boissonnat, André Lieutier, and Mathijs Wintraecken. The reach, metric distortion, geodesic convexity and the variation of tangent spaces. *Journal of Applied and Computational Topology*, 3(1):29–58, Jun 2019.
3. Frédéric Chazal, Vin De Silva, Marc Glisse, and Steve Oudot. *The structure and stability of persistence modules*, volume 10. Springer, 2016.
4. Éric Colin de Verdière. Computational topology of graphs on surfaces. In Jacob E. Goodman, Joseph O'Rourke, and Csaba Toth, editors, *Handbook of Discrete and Computational Geometry*, chapter 23, pages 605–636. CRC Press LLC, third edition, 2018.
5. Tamal K. Dey, Herbert Edelsbrunner, and Sumanta Guha. Computational topology. In Bernard Chazelle, Jacob E. Goodman, and Richard Pollack, editors, *Advances in Discrete and Computational Geometry – Proc. 1996 AMS-IMS-SIAM Joint Summer Research Conf. Discrete and Computational Geometry: Ten Years Later*, number 223 in Contemporary Mathematics, pages 109–143. AMS, 1999.
6. Herbert Federer. Curvature measures. *Transactions of the American Mathematical Society*, 93:418–491, 1959.
7. Jacob E. Goodman, Joseph O'Rourke, and Csaba Tóth, editors. *Handbook of discrete and computational geometry*. CRC Press LLC, Boca Raton, FL, third edition, 2018.
8. Larry Guth. *Polynomial methods in combinatorics*, volume 64. American Mathematical Soc., 2016.
9. Thomas C. Hales. A proof of the kepler conjecture. *Annals of mathematics*, pages 1065–1185, 2005.
10. Joel Hass, Jeffrey C. Lagarias, and Nicholas Pippenger. The computational complexity of knot and link problems. *Journal of the ACM (JACM)*, 46(2):185–211, 1999.
11. Jun Kitagawa, Quentin Mérigot, and Boris Thibert. Convergence of a newton algorithm for semi-discrete optimal transport. *Journal of the European Mathematical Society*, 21(9):2603–2651, 2019.
12. Greg Kuperberg. How hard is it to approximate the Jones polynomial? *Theory Comput.*, 11(1):183–219, 2015.
13. Marc Lackenby. Algorithms in 3-manifold theory. *Surveys in Differential Geometry*, 25, 2022.
14. Francis Lazarus, Michel Pocchiola, Gert Vegter, and Anne Verroust. Computing a canonical polygonal schema of an orientable triangulated surface. In *Proceedings of the 17th Annual Symposium on Computational Geometry (SOCG)*, pages 80–89. ACM, 2001.
15. Jacob Leygonie, Steve Oudot, and Ulrike Tillmann. A framework for differential calculus on persistence barcodes. *Foundations of Computational Mathematics*, 22(4):1069–1131, 2022.
16. Partha Niyogi, Stephen Smale, and Shmuel Weinberger. Finding the homology of submanifolds with high confidence from random samples. *Discrete & Computational Geometry*, 39(1-3):419–441, 2008.

17 Marcus Schaefer, Jean Cardinal, and Tillmann Miltzow. The existential theory of the reals as a complexity class: A compendium. *arXiv preprint arXiv:2407.18006*, 2024.

Twenty Years of GdR IFM, seen from GT Discrete Geometry and Mathematical Morphology

The overview of the working group on *Géométrie Discrète et Morphologogie Mathématique* (GDMM) is structured around five major scientific axes developed continuously from 2005 to 2025: discrete geometric objects, interactions in combinatorial optimization, topology for imaging, watershed transforms for segmentation, and morphological filtering. These axes are complemented by a transversal axis devoted to software development and open, reproducible science.

1 Discrete geometry: multigrid convergence and discrete calculus

Discrete geometry, also known as digital geometry, is primarily concerned with defining a consistent geometry on subsets of the discrete grid $\mathbb{Z}^d$, or more generally on lattices or tilings. It was originally studied within the framework of the geometry of numbers in mathematics, with strong links to arithmetic and convex geometry. Since the 1970s, the scope of discrete geometry has been profoundly influenced and transformed by the emergence of digital images—first in 2D, then rapidly in 3D with computed tomography and nuclear magnetic resonance imaging. The subsets under study have become discretizations of real physical shapes, or even a means of efficiently encoding the geometry of virtual shapes. Questions that were initially purely mathematical have thus been reformulated in more applied terms, with on the one hand the constraint of developing a geometry capable of addressing concrete problems (such as measuring length or area, estimating tangents, normals, or curvatures, identifying smooth or convex regions, corners, or edge lines), and on the other hand algorithmic constraints, such as decidability or efficiency in time and memory.

Whereas research conducted in the 1980s–2000s was essentially focused on the study of linear/affine geometry—particularly the definition and recognition of discrete objects (lines, planes, circles) using arithmetic or combinatorial methods—the last twenty years have been marked by the study of the asymptotic consistency of discrete geometry results with those of Euclidean geometry. The property of **multigrid convergence** of a geometric estimator precisely guarantees that a finer discretization of a Euclidean object leads to a more accurate geometric estimation based solely on the discretized object. Gauss had already shown that counting the number of integer points inside a convex volume provides a good approximation of its volume. While multigrid convergence of estimators for perimeter, length, or moments was already known in the early 2000s (see, e.g., [28]), it was only in the following decade that multigrid convergence of local geometric estimators, such as tangents or normals, was established, using a wide variety of approaches in 2D: maximal segments [19, 35], binomial convolutions [21], and polynomial approximations [50]. Approaches for 3D surfaces have also been proposed, such as binomial convolutions [22] or planarity probing [31], though with limited guarantees. This was followed by the development of "integral" or "geometric measure theory" approaches, which estimate differential geometric quantities via carefully chosen integrals, leading to more robust methods [13, 17, 42]. The current theoretical state of the art is achieved by normals computed using integral invariants [30], which also enable the estimation of all curvatures [13]. These convergent normals induce an area estimator on discretized surfaces in any dimension, and more generally the convergence of any surface integral [34]. Since 2021, the theory of corrected normal currents has been the best approach for curvature estimation [32]. Initially developed for digital surfaces (see Figure 1), its theoretical framework and stability also make it the state of the art for polygonal surfaces

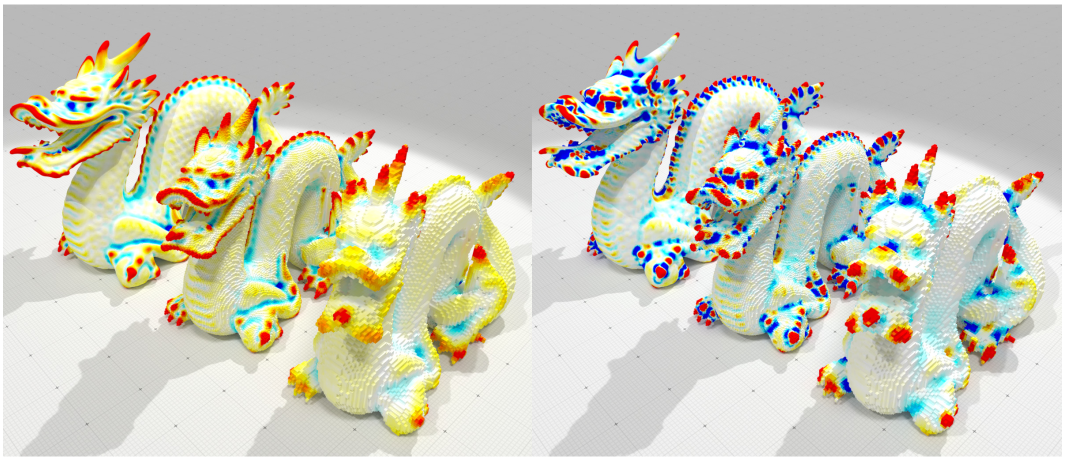


Figure 1 Convergent estimation of mean (left) and Gaussian (right) curvatures on digital surfaces [32], implemented in the DGtal library.

[33] and point clouds [29].

The development of a convergent discrete differential geometry has fostered the emergence of **discrete exterior calculus** on digital surfaces over the past decade. Discrete exterior calculus was initially approached through methods originating from theoretical physics [41] or from graph theory and electrical networks [26]. To obtain convergence properties, methods developed for triangulated or polygonal surfaces [18, 20] were adapted to the case of digital surfaces. The incorporation of convergent estimators of normal vectors into this calculus induces the convergence of classical differential operators, such as the gradient, the Laplacian, or the divergence [9, 53]. This framework then enables the geometric processing of digital data via these operators (e.g., spectral decomposition of the Laplacian, piecewise-smooth regularization of Mumford–Shah type, geodesics, UV parameterization).

2 Combinatorial optimization and discrete geometry

The founders of digital geometry envisioned a new computational paradigm suited to numerical data modeled as sets of pixels or voxels. The initial idea of providing a fully discrete computational model for data that are themselves perfectly digital has given way to a scientific environment in which discrete and continuous tools have come together and combined to address increasingly complex problems.

Hermann Minkowski's geometry of numbers, the study of polytopes with integer vertices, combinatorial optimization, geometric optimization, algorithmic geometry, combinatorial geometry, and discrete geometry have not diverged but, on the contrary, have converged toward a set of questions that do not constitute isolated compartments but rather a continuum of problems that are both combinatorial and geometric.

Let us take the example of Multi-Agent Path Finding (MAPF), which involves navigating a fleet of robots moving on a grid, as seen in competitions such as the League of Robot Runners or CG:SHOP 2021. Digital geometry is fundamental here: the robots operate in a discretized space, their trajectories are combinatorial, yet the search for a global optimum (minimizing total time, avoiding collisions, respecting spatial constraints) fully falls under geometric and algorithmic optimization. Such problems perfectly illustrate the convergence of the two fields: a discrete numerical geometric space tackled using all possible continuous or discrete optimization methods—SAT solvers, MIP, graph- and geometry-based metaheuristics, or even Deep Learning.

In the age of combinatorial games on smartphones, many games—from the old tic-tac-toe to Tetris, Numberlink, Block Blast, and Flow Free—take place on a 2D grid. The combinatorial richness of the grid has fascinated mathematicians since ancient times: from Euler's Latin squares to John Conway's Game of Life, it has become a favorite testing ground for algorithmic complexity. Geometric problems such as tiling and discrete tomography are examples of this. The last twenty years have seen an explosion of results on the decidability, complexity, and approximation of geometric problems on grids, where the boundaries between combinatorics, optimization, and geometry are blurred.

Digital geometry has also greatly benefited from these interactions. For example, determining whether a subset S of $\mathbb{Z}^2$ is digitally convex can be decided in linear time. The problem of covering S, assumed to be hole-free, with a minimal number of balls (not containing any integer point outside S) can be solved in polynomial time, whereas these problems are typically hard in classical computational geometry. The convex skull problem can be solved digitally in $O(k^3)$ time for a set of k integer points, whereas for a polygon with n vertices, the complexity is $O(n^7)$. This demonstrates that classical problems in computational geometry change character when considered in the framework of digital geometry, leading to original theoretical developments. Nevertheless, there are strong similarities, one of which is the central role played by convexity, which is itself the subject of new developments. The computational geometry community has also taken up combinatorial problems from digital geometry, such as polyomino folding or polycube unfolding [51]. The french community is very active on these topics, as illustrated by the 1st place for the team "*Les Shadocks*" in the CG:SHOP (Computational Geometry: Solving Hard Optimization Problems) challenge in 2021, 2022, 2024 and 2026.

Today, the boundary between digital geometry, algorithmics, and optimization has never been more porous. Deep learning introduces yet another dimension: geometric machine learning models learn to reconstruct, simplify, or parameterize surfaces directly from numerical data. This hybridization of the discrete, continuous and statistical spaces opens a new era in which numerical geometry becomes simultaneously a tool for analysis, optimization, representation, and modeling.

3 Topology for imaging

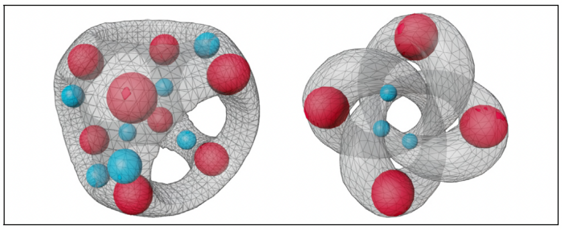
Following the example of discrete geometry, whose main goal is to develop concepts in Cartesian spaces ($\mathbb{Z}^d$) that are consistent with Euclidean geometry ($\mathbb{R}^d$), in the 1960s work was also initiated to develop topological frameworks adapted to digital imaging and dedicated to coherently modeling the topology of the continuous scenes underlying images. The historical work carried out during the first decades, in conjunction with methodological and technological advances in 2D and then 3D digital imaging, initially focused on the development of (1) discrete topological frameworks based on $\mathbb{Z}^d$ (digital topology, cubic complexes); (2) discrete versions of classical topological invariants; (3) operational tools for the development of topology-based transformations (e.g., skeletonization), under the impetus of pioneers such as Rosenfeld, Pfaltz, Khalimsky, Kovalevsky... Until the turn of the 2000s, this work was mainly dedicated to binary images.

During the period 2005-2025, the French community, within the GT GDMM, contributed to major advances in this field, with the extension of concepts in terms of image dimensions (3D, 4D) and valuation (grayscale images, labeled images), but also through the generalization of notions and the creation of new links with continuous topology and connective frameworks developed in mathematical morphology.

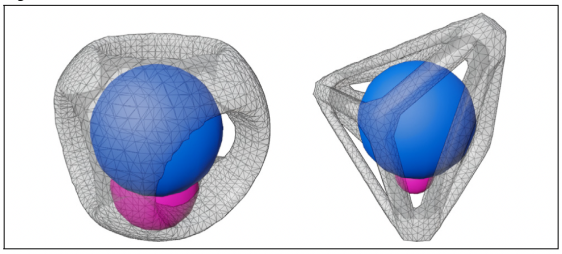
The main contributions of the first decade (2005-2015) concerned work on homotopic transformations, led by researchers such as Bertrand, Couprie… In particular, the historical notion of "simple point", widely studied in the context of binary 3D digital topology during the 1990s, was extended in multiple ways: in dimension (4D) [15], in valuation (grayscale, labels [40]), in cardinality (simple sets). The extension of the notion of simple points from digital topology to complex models has also led to the proposal of more general concepts of homotopic transformations in complexes, notably with the proposal of critical kernels [5], which provide guarantees of topological and geometric optimality (directly related to median axes) for objects obtained by decreasing homotopic transformations, particularly in parallel paradigms. Another notable contribution during the same period concerns the formal proof of the compatibility of discrete and continuous topology frameworks [39].

During the second decade (2015-2025), the spectrum of contributions became much more diverse. Without claiming to be exhaustive, the following advances are particularly noteworthy. (a) The conditions for preserving topological properties during the continuous-discrete transition, initially explored in the mid-1980s, were extended [36] to also consider alternative paradigms (e.g., discrete-discrete in the context of "re-digitization"), leading in particular to homotopic preservation guarantees for geometric transformations [44]. (b) The notion of a "well-composed set" proposed in the 1990s has been extensively developed and consolidated in different frameworks and dimensions, making it possible, on the one hand, to provide dimensional guarantees on the objects manipulated [7], but also to develop sufficient conditions for symmetrically managing the open/closed paradigm, leading in particular to the proposal of truly self-dual hierarchical structures for image modeling [6]. (c) An axiomatic framework based on completions, which allows several key concepts (simple homotopy, homology, etc.) to be formalized inductively and related to each other, has been proposed in the context of simplicial complexes (obviously related to cubic complexes) [2].

Although continuous efforts were made during this period to develop effective concepts and algorithms for calculating homology groups and related descriptors [25, 46] (e.g., Betti numbers), the recent democratization of topological data analysis (TDA), as well as the advent of deep learning (DL) and the relevance of integrating topological priors into their model, have (re)motivated the development of sustained research on topological invariants. The most recent work, initiated at the end of the period and offering prospects for the coming years, focuses in particular on the notion of persistence accessible in hierarchical structures (morphological trees) modeling digital images, which echoes the persistence implemented in TDA. In this context, the construction of topological loss functions based on trees [48], the integration of binary invariants into these trees resulting in grayscale invariants [45], or the links between the notion of dynamics and homological persistence [8], are all avenues of research that could consolidate the bridges between discrete topology, mathematical morphology, and TDA in the coming years. Finally, we can cite very recent work that links discrete Morse theory and the watershed framework [4] and, more generally, the notions of "Morse sequences" and "Morse frames" [3], which revisit discrete Morse theory and pave the way for efficient computation of persistence in homology and cohomology in the framework of simplicial complexes.



(a) hole-balls of 0-holes (including non-present holes) for *hollow-cubic-ball* and *knot*. Light blue balls are B-balls with $B < 0$.



(b) hole-balls of non-present 2-holes for *hollow-cubic-ball* and *triakis-tetrahedron*. Magenta balls are T-balls with $T < 0$.

Figure 2 Calculation of cavity measurements in volumetric objects defined by surface meshes. Illustration based on [23].

4 Watershed and image segmentation

The concept of the watershed, studied since the 19th century by scientists such as Camille Jordan and James Clerk Maxwell, describes how a landscape can be divided into catchment basins, each associated with a minimum toward which water flows. The watershed line separates these basins and corresponds to locations where water can flow toward different minima. In the late 1970s, this idea was adapted to image processing by interpreting an image as a topographic surface where pixel intensity represents altitude, allowing images to be segmented into coherent regions. Efficient algorithms were developed in the 1990s and incorporated into many software tools, including medical imaging and photo editing. However, in 2005 it was shown that previous definitions either rely on unrealistic continuity assumptions for digital images or fail to satisfy certain desirable properties.

To overcome earlier limitations, the concept of the watershed was redefined using edge-weighted graphs [16]. This new definition, based on the intuitive drop of water principle, satisfies important properties that previous discrete definitions lacked. In particular, it establishes a duality between catchment basins defined by steepest-descent paths and their separation defined by flow divergence. The key result shows that watersheds correspond to cuts of minimum spanning forests rooted at the graph's minima, linking the concept to a classical problem in combinatorial optimization and creating connections with algorithms, topology, morphological filtering, and hierarchical analysis.

For example, a new generic combinatorial optimization problem on graphs parameterized by two exponent values p and q, which allows several known methods to be unified, has been defined and studied. Depending on the values of p and q, the solution is a minimal cut, a random walker cut, a Voronoi diagram, or finally, when p tends to infinity, a new combinatorial object called a power watershed [14]. The power watershed is a minimum spanning forest cut (thus, a watershed cut) for which certain forests of the same weight are discriminated by a secondary optimality criterion. This result links several methods that are useful in image analysis and combines their advantages.

In practice, the images and data contain different levels of detail whose importance varies depending on the application. It is therefore appropriate to consider a hierarchical representation in which watershed basins are progressively merged as one moves up the scale levels. To this end, a filtering process, known as connected closing, is considered. It consists of "filling in" certain minima according to a relevance criterion and the desired degree of severity. By varying the severity, a series of filtered images and their corresponding watershed lines are generated. Other types of hierarchies are studied within the community: component trees, trees of shapes, binary partition trees, quasi-flat zones and alpha trees, as well as links with ultrametric distances and hierarchical clustering theory, such as single-linkage clustering or the HDBSCAN method. The international state of the art in image segmentation generally includes a hierarchical segmentation step, most often obtained using the watershed transform. The most recent work in our community consists of integrating segmentation algorithms into deep learning models in order to predict segmentations [12].

This methodological development was accompanied by algorithmic development based on schemes such as Kruskal, Prim, or Boruvka algorithms, leading to improvements in the complexity and efficiency in the computation of these various structures. Several scenarios were studied, leading to interactive, distributed, parallel, out-of-core algorithms capable of handling large data (gigapixels) and integrating the developed operators on GPUs, making them easier to integrate into deep learning chains.

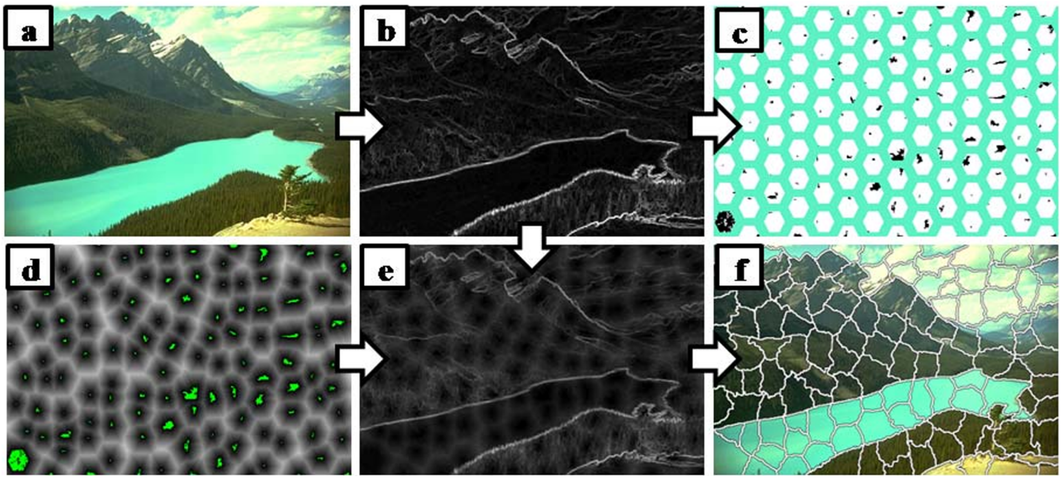


Figure 3 Illustration of the waterpixel method, which segments the image (see f) into regular regions distributed across the image, with contours that adhere to those of the image. To do this, a watershed is calculated from the image (e) obtained by combining the contours (b) of the original image (a) with a Voronoi diagram (d) of the minima positioned regularly throughout the image (c). Illustration based on [38].

5 Morphological filtering

Mathematical morphology is known worldwide as a mathematically well-founded technique and theory, inspired by image processing problems, which constitute its main field of application. The theory of morphological filtering is based on the algebraic structure of lattices. The elementary operators, dilation and erosion, are defined as applications of one lattice to another (or to itself) that commute with the supremum and infimum, respectively, forming pairs called adjunctions. Many more complex filters are constructed by composing these basic operators. Unlike classical image processing operators, which are generally linear and invertible, mathematical morphology operators are nonlinear, non-invertible, and have specific properties such as idempotence and growth. This characteristic results in a controlled loss of information that leads to powerful and targeted filtering. In particular, these operators can be used to remove structures that do not meet certain geometric criteria, such as width, volume, or contrast. Over the past twenty years, the French community has contributed significantly to the evolution of this field by extending conventional operators to more complex frameworks that are better suited to modern applications.

In practice, mathematical morphology operators are often parameterized by a structuring element, i.e., a shape used to guide the effect of the operator in its definition space, which is generally a discrete grid with integer coordinates. On the one hand, spatially variant and adaptive morphological operators have been developed. These operators allow the shape of the structuring element to be varied according to its position in space. This makes it possible to adapt the filtering to the content of an image or image area [37]. This may involve varying the direction of a linear structuring element to filter fine structures such as a vascular network, or reducing the regularization effect in high-contrast areas of the image [43]. On the other hand, the study of morphological operators in spaces of graphs, hypergraphs, simplicial complexes, point clouds, fuzzy and bipolar sets, or even logical relations is a very active field. The goal is, for example, to perform fine-grained operations on the topology of objects or to formalize spatial relations such as "is to the right of", "is located between", "is intertwined with", or even to detect certain patterns in pieces of music.

Another category of operators, called connected operators, acts solely by removing or preserving connected components of the object or its complement. An important property of these operators is that they can only remove contours: creating new contours or moving existing contours is not possible. This leads to filtering that does not blur, unlike standard operators. The selection of components to be removed is based on geometric or photometric attributes. To effectively represent these components and evaluate their attributes, images are represented by hierarchical structures that encode the inclusion of related components. Component trees (directed [49] or undirected [10]) or level set trees [24] are among the most studied in recent decades. Our community has proposed increasingly efficient algorithms for calculating and processing them.

Regardless of the structure of the definition space, another interesting area of research concerns the processing of non-scalar images. In this case, there is no natural order to the values. A choice must be made to define a lattice suited to the nature of the data and the application objectives of the processing. This theoretical challenge has given rise to numerous proposals, particularly for color imaging (as opposed to binary or grayscale imaging) [1], multi- or hyperspectral imaging (as in astronomy or remote sensing) [52], and even tensor imaging, available in certain MRI modalities.

Finally, the most recent work aims to integrate mathematical morphology operators into deep learning models. This involves, for example, inferring the shape of the structural elements to be used to perform a given task [27], or determining whether a neural network model can be more effective by replacing certain convolutional layers with mathematical morphology operators [47]. Another challenge is to train a neural network to produce results with desired topological characteristics: number of desired minima or maxima, removal of connected components corresponding to false positives, or reconnection of true positives [48].

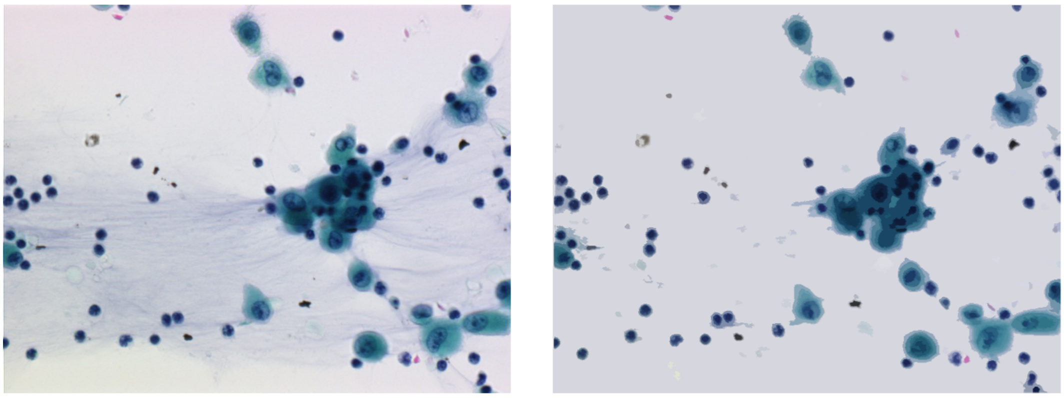


Figure 4 Simplification (right) of a microscopic color image of the bronchi (left) in cytology: connected filtering removes non-circular shapes in order to highlight the nuclei and cytoplasm. Illustration based on [11].

6 Reproducible research and software development

The GDMM community is committed to reproducible research through the provision of open codes and data. Among the first initiatives was the development of the Pink library. It contains more than 400 image processing algorithms and operators, many of which originate from the work of the GDMM community as a whole. We can also mention the development of the OLENA-MILENA-PYLENE library, which focuses on a software genericity approach: the same code must work for "all" possible data types. For image processing, we can also mention the Higra library, which is a reference for hierarchical analysis of images and data in mathematical morphology and is capable of interfacing with modern deep learning tools, as is MorphoLayers dedicated to this interaction between mathematical morphology and deep learning, which uses Keras/Tensorflow for this purpose.

The DGtal library is a collaborative effort which brings together the main algorithms from French research in discrete geometry. It is currently the benchmark and the state-of-the-art library in digital geometry, with more than 400,000 lines of code, 160 pages of documentation and hundreds of forks. It is developed in C++ and includes data-structures, algorithms, tools and interfaces for processing and visualizing images and geometric objects in arbitrary dimension, with also a front-end in Python. DGtal received the *software award* at the *Symposium on Geometry Processing* in 2016.

To complete this overview, we would like to highlight the efforts of the GDMM community to structure and promote reproducible research. In particular, the GDMM community is active in the editorial board of the journal IPOL (Image Processing OnLine), in which each article contains a text on an algorithm and its source code with an online demonstrator and an archive of the experiments carried out. It also helped establishing the RRPR (Reproducible Research in Pattern Recognition) workshop in the international image analysis landscape. This workshop has been held every two years since 2016 during the ICPR international conference (a major event organized by the International Association for Pattern Recognition, IAPR). Since 2024, RRPR has been a full-fledged technical committee of the IAPR (designated TC-22), with a vice-chair from the GDMM community, while discrete geometry and mathematical morphology form TC-18.

Contributors.

Jean Cousty, Yan Gerard, Jacques-Olivier Lachaud, Phuc Ngo, Nicolas Passat, and Isabelle Sivignon.

References

1 Jesús Angulo. Morphological colour operators in totally ordered lattices based on distances: Application to image filtering, enhancement and analysis. *Computer Vision and Image Understanding*, 107(1):56–73, 2007.

2 Gilles Bertrand. Completions, perforations and fillings. In *DGMM, Proceedings*, pages 137–151, 2021.

3 Gilles Bertrand. Morse sequences: A simple approach to discrete Morse theory. *Journal of Mathematical Imaging and Vision*, 67:16, 2025.

4 Gilles Bertrand, Nicolas Boutry, and Laurent Najman. Discrete Morse functions and watersheds. *Journal of Mathematical Imaging and Vision*, 65:787–801, 2023.

5 Gilles Bertrand and Michel Couprie. On parallel thinning algorithms: Minimal non-simple sets, P-simple points and critical kernels. *Journal of Mathematical Imaging and Vision*, 35:23–35, 2009.

6 Nicolas Boutry, Thierry Géraud, and Laurent Najman. How to make n-D plain maps defined on discrete surfaces alexandrov-well-composed in a self-dual way. *Journal of Mathematical Imaging and Vision*, 61:849–873, 2019.

7 Nicolas Boutry, Laurent Najman, and Thierry Géraud. Equivalence between digital well-composedness and well-composedness in the sense of Alexandrov on n-d cubical grids. *Journal of Mathematical Imaging and Vision*, 62:1285–1333, 2020.

8 Nicolas Boutry, Laurent Najman, and Thierry Géraud. Some equivalence relation between persistent homology and morphological dynamics. *Journal of Mathematical Imaging and Vision*, 64:807–824, 2022.

9 Thomas Caissard, David Coeurjolly, Jacques-Olivier Lachaud, and Tristan Roussillon. Laplace–Beltrami operator on digital surfaces. *Journal of Mathematical Imaging and Vision*, 61(3):359–379, 2019.

10 Edwin Carlinet and Thierry Géraud. A comparative review of component tree computation algorithms. *IEEE TIP*, 23(9):3885–3895, 2014.

11 Edwin Carlinet and Thierry Géraud. Mtos: A tree of shapes for multivariate images. *IEEE Transactions on Image Processing*, 24(12):5330–5342, 2015.

12 Giovanni Chierchia and Benjamin Perret. Ultrametric fitting by gradient descent. *Advances in Neural Information Processing Systems*, 32, 2019.

13 David Coeurjolly, Jacques-Olivier Lachaud, and Jérémy Levallois. Multigrid convergent principal curvature estimators in digital geometry. *Computer Vision and Image Understanding*, 129:27–41, 2014.

14 Camille Couprie, Leo Grady, Laurent Najman, and Hugues Talbot. Power watershed: A unifying graph-based optimization framework. *IEEE Transactions on Pattern Analysis and Machine Intelligence (TPAMI)*, 33(7):1384–1399, 2011.

15 Michel Couprie and Gilles Bertrand. New characterizations of simple points in 2D, 3D, and 4D discrete spaces. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 31:637–648, 2009.

16 Jean Cousty, Gilles Bertrand, Laurent Najman, and Michel Couprie. Watershed cuts: Minimum spanning forests and the drop of water principle. *IEEE TPAMI*, 31(8):1362–1374, 2009.

17 Louis Cuel, Jacques-Olivier Lachaud, Quentin Mérigot, and Boris Thibert. Robust geometry estimation using the generalized voronoi covariance measure. *SIAM Journal on Imaging Sciences*, 8(2):1293–1314, 2015.

18 Fernando De Goes, Andrew Butts, and Mathieu Desbrun. Discrete differential operators on polygonal meshes. *ACM Transactions on Graphics (TOG)*, 39(4):110–1, 2020.

19 François De Vieilleville, Jacques-Olivier Lachaud, and Fabien Feschet. Convex digital polygons, maximal digital straight segments and convergence of discrete geometric estimators. *Journal of Mathematical Imaging and Vision*, 27(2):139–156, 2007.

20 Mathieu Desbrun, Anil N Hirani, Melvin Leok, and Jerrold E Marsden. Discrete exterior calculus. *arXiv preprint math/0508341*, 2005.

21 Henri-Alex Esbelin, Rémy Malgouyres, and Colin Cartade. Convergence of binomial-based derivative estimation for c2 noisy discretized curves. *Theoretical Computer Science*, 412(36):4805–4813, 2011.

22 Sébastien Fourey and Rémy Malgouyres. Normals estimation for digital surfaces based on convolutions. *Computers & Graphics*, 33(1):2–10, 2009.

23 Yann-Situ Gazull, Alexandra Bac, and Aldo Gonzalez-Lorenzo. Computing geometrical measures of topological holes. *Computer-Aided Design*, 163:103563, 2023.

24 Thierry Géraud, Edwin Carlinet, Sébastien Crozet, and Laurent Najman. A quasi-linear algorithm to compute the tree of shapes of n d images. In *International symposium on mathematical morphology and its applications to signal and image processing*, pages 98–110, 2013.

25 Aldo Gonzalez-Lorenzo. *Computational Homology Applied to Discrete Objects*. PhD thesis, Aix-Marseille University, France, 2016.

26 Leo J Grady and Jonathan R Polimeni. *Discrete calculus: Applied analysis on graphs for computational science*, volume 3. Springer, 2010.

27 Romain Hermary, Guillaume Tochon, Élodie Puybareau, Alexandre Kirszenberg, and Jesús Angulo. Learning grayscale mathematical morphology with smooth morphological layers. *Journal of Mathematical Imaging and Vision*, 64(7):736–753, 2022.

28 Reinhard Klette and Azriel Rosenfeld. *Digital geometry - geometric methods for digital picture analysis*. Morgan Kaufmann, 2004.

29 Jacques-Olivier Lachaud, David Coeurjolly, Céline Labart, Pascal Romon, and Boris Thibert. Lightweight curvature estimation on point clouds with randomized corrected curvature measures. *Computer Graphics Forum*, 42(5):e14910, 2023.

30 Jacques-Olivier Lachaud, David Coeurjolly, and Jérémy Levallois. Robust and convergent curvature and normal estimators with digital integral invariants. In *Modern Approaches to Discrete Curvature*, pages 293–348. Springer, 2017.

31 Jacques-Olivier Lachaud, Xavier Provençal, and Tristan Roussillon. Two plane-probing algorithms for the computation of the normal vector to a digital plane. *Journal of Mathematical Imaging and Vision*, 59(1):23–39, 2017.

32 Jacques-Olivier Lachaud, Pascal Romon, and Boris Thibert. Corrected curvature measures. *Discrete & Computational Geometry*, 68(2):477–524, 2022.

33 Jacques-Olivier Lachaud, Pascal Romon, Boris Thibert, and David Coeurjolly. Interpolated corrected curvature measures for polygonal surfaces. *Computer Graphics Forum*, 39(5):41–54, 2020.

34 Jacques-Olivier Lachaud and Boris Thibert. Properties of gauss digitized shapes and digital surface integration. *Journal of Mathematical Imaging and Vision*, 54(2):162–180, 2016.

35 Jacques-Olivier Lachaud, Anne Vialard, and François de Vieilleville. Fast, accurate and convergent tangent estimation on digital contours. *Image and Vision Computing*, 25(10):1572–1587, 2007.

36 Étienne Le Quentrec, Loïc Mazo, Étienne Baudrier, and Mohamed Tajine. Local turn-boundedness: A curvature control for continuous curves with application to digitization. *Journal of Mathematical Imaging and Vision*, 62:673–692, 2020.

37 Romain Lerallut, Étienne Decencière, and Fernand Meyer. Image filtering using morphological amoebas. *Image and Vision Computing*, 25(4):395–404, 2007.

38 Vaïa Machairas, Matthieu Faessel, David Cárdenas-Peña, Théodore Chabardes, Thomas Walter, and Etienne Decenciere. Waterpixels. *IEEE Transactions on Image Processing*, 24(11):3707–3716, 2015.

39 Loïc Mazo, Nicolas Passat, Michel Couprie, and Christian Ronse. Digital imaging: A unified topological framework. *Journal of Mathematical Imaging and Vision*, 44:19–37, 2012.

40 Loïc Mazo, Nicolas Passat, Michel Couprie, and Christian Ronse. Topology on digital label images. *Journal of Mathematical Imaging and Vision*, 44:254–281, 2012.

41 Christian Mercat. Discrete Riemann surfaces and the Ising model. *Communications in Mathematical Physics*, 218(1):177–216, 2001.

42 Quentin Mérigot, Maks Ovsjanikov, and Leonidas J Guibas. Voronoi-based curvature and feature estimation from point clouds. *IEEE Transactions on Visualization and Computer Graphics*, 17(6):743–756, 2010.

43 Odyssée Merveille, Hugues Talbot, Laurent Najman, and Nicolas Passat. Curvilinear structure analysis by ranking the orientation responses of path operators. *IEEE Transactions on Pattern Analysis and Machine Intelligence (TPAMI)*, 40(2):304–317, 2018.

44 Phuc Ngo, Nicolas Passat, Yukiko Kenmochi, and Hugues Talbot. Topology-preserving rigid transformation of 2d digital images. *IEEE Transactions on Image Processing*, 23:885–897, 2014.

45 Nicolas Passat, Julien Mendes Forte, and Yukiko Kenmochi. Morphological hierarchies: A unifying framework with new trees. *Journal of Mathematical Imaging and Vision*, 65:718–753, 2023.

46 Samuel Peltier, Sylvie Alayrangues, Laurent Fuchs, and Jacques-Olivier Lachaud. Computation of homology groups and generators. *Computers & Graphics*, 30:62–69, 2006.

47 Valentin Penaud–Polge, Santiago Velasco-Forero, and Jesus G. Angulo. Group equivariant morphological networks. *SIAM Journal on Imaging Sciences*, 18(4):2236–2276, 2025.

48 Benjamin Perret and Jean Cousty. Component tree loss function: Definition and optimization. In *International Conference on Discrete Geometry and Mathematical Morphology*, pages 248–260, 2022.

49 Benjamin Perret, Jean Cousty, Olena Tankyevych, Hugues Talbot, and Nicolas Passat. Directed connected operators: Asymmetric hierarchies for image filtering and segmentation. *IEEE TPAMI*, 37(6):1162–1176, 2014.

50 Laurent Provot and Yan Gérard. Estimation of the derivatives of a digital function with a convergent bounded error. In *International Conference on Discrete Geometry for Computer Imagery*, pages 284–295, 2011.

51 Lydie Richaume, Eric Andres, Gaëlle Largeteau-Skapin, and Rita Zrour. Unfolding h-convex manhattan towers. *J. Comb. Optim.*, 44(4):3023–3037, 2022.

52 Santiago Velasco-Forero and Jesus Angulo. Supervised ordering in $\mathbb{R}^p$: Application to morphological processing of hyperspectral images. *IEEE TIP*, 20(11):3301–3308, 2011.

53 Colin Weill-Duflos, David Coeurjolly, and Jacques-Olivier Lachaud. Corrected Laplace–Beltrami operators for digital surfaces. *Journal of Mathematical Imaging and Vision*, 67(2):11, 2025.

Les vingt ans du GdR IFM, vus du GT « Modélisation géométrique »

Le début de notre communauté remonte à 1995 en tant que groupe de travail du GDR-PRC AMI (cadre du pôle Informatique Graphique) et de l'AFIG. Le GDR AMI a cessé d'exister fin 1997 et a cédé la place au GDR ALP (Algorithmique, Langages, Programmation), avec un pôle Informatique graphique et le groupe de travail Modélisation Géométrique s'est poursuivi. Depuis 2006, le GTMG fait partie du GDR IM (Informatique Mathématique, puis IFM en 2024) et du GDR IG (puis IGRV en 2014). Le GT travaille sur la définition de modèles géométriques, la description d'objets en tant que courbes et surfaces paramétriques, maillages, nuages de points, voxels et volumes. Il s'intéresse donc à la description, aux traitements, à la caractérisation de modèles discrets, continus ou semi-continus, ponctuels, linéiques, surfaciques, volumiques en dimension 2, 3 ou plus. La transformation entre les modèles est également un enjeu important, ainsi que la cohérence topologique de ceux-ci.

Le GT reste proche de l'application de ses travaux, en spécialisant modèles et traitements. Parmi les applications, la production industrielle (modélisation, façonnage, analyse) à toujours pris une place importante, ce qui explique le lien avec la communauté SMAI-AFA et les écoles d'ingénieurs comme l'ENSAM. L'aspect créatif et artistique des objets produits, à une échelle moindre, est toujours actif dans le groupe, on peut citer notamment les travaux des laboratoires LJK, ACROE, CRESTIC.

1 Topologie

La topologie est un point important du groupe, car la cohérence des modèles se base également sur ces concepts. On peut citer les travaux sur les cartes généralisées (XLIM). Les propriétés topologiques, les groupes d'homologie, le contrôle d'entités topologiques pendant la construction d'un objet prennent de l'importance depuis une dizaine d'années (ICUBE, XLIM, LIS, Figure 1). Les cartes combinatoires multirésolution décrivent des maillages de dimension quelconque. Des opérateurs topologiques et géométriques permettent de travailler d'abord en dimension 3 sur des maillages tétraédriques et hexaédriques, puis plus généralement sur des topologies arbitraires avec une approche multi-échelle. Les membres du groupe développent aussi une grande expertise en analyse topologique de données à travers des travaux sur la réduction de dimension de nuages de points par exemple. Finalement l'extraction de caractéristiques topologiques, comme l'axe médian reste un sujet très actif (Figure 1).

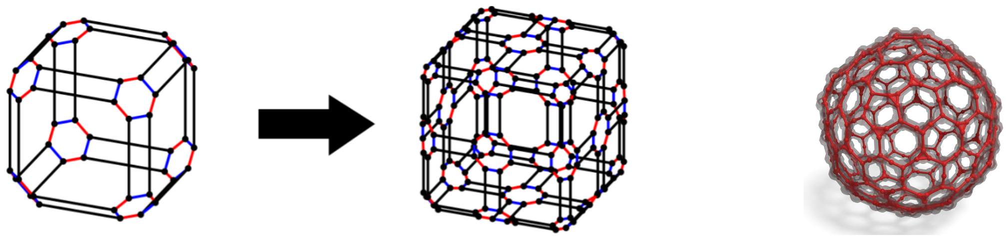


Figure 1 À gauche : Règles de subdivision pour un cube représenté sous forme de G-cartes (Repris de [24]. À droite : Extraction de l'axe médian d'une forme (Repris de [7]).

2 Reconstruction

La reconstruction ou la création d'un lien entre des espaces de dimensions différentes est généralement une transformation de modèles géométriques. Cela couvre les opérations classiques de triangulation 2D et 3D (sujet proche des thématiques du GT GDMM), de remaillage (INRIA Sophia-Antipolis) et a pris une force importante lors de la généralisation de la production des nuages de points (scanner laser, photogrammétrie) dans les années 2000. Les applications ont permis de définir de nouveaux traitements notamment en production industrielle (Scan2CAD, Scan2BIM, équipe du LISPEN) afin de filtrer, caractériser et annoter sémantiquement les données produites. Rapidement, les traitements se sont portés sur les nuages de points directement (*i.e.* sans voisinage, sans notion de topologie) afin d'éviter l'étape de maillage (consommatrice en temps, erreurs et espace, et qui constitue en soi une première interprétation de la donnée brute acquise). On peut citer les travaux du LIRIS ou de l'IRIT. Il est apparu également l'idée de joindre la géométrie à d'autres modalités comme l'intensité de laser, ou la couleur, permettant de spécialiser ou de lever des ambiguïtés dans les traitements (LIRIS). Cette thématique a fortement bénéficié du bond de l'IA à partir de 2010.

3 Caractérisation

La caractérisation est une thématique importante pour le retour d'information vers l'expertise de plus haut niveau. Elle a été motrice des algorithmes de détection, de segmentation, d'extraction de formes (*features*).

L'IA a un impact important sur la caractérisation d'objets : les processus de segmentation construits précédemment par analyses locale et globale, par référence à un dictionnaire de formes, par des systèmes experts ont pu être remplacés par des modèles supervisés ou non. Cela a rapidement posé à la communauté le problème de la création de modèles pour l'entraînement, ce qui reprend les règles de construction (CAD, architecture, mécanique). Cela permet également de comparer des objets 3D sous forme de graphes de propriétés géométriques et topologiques (GNN - *Graph neural network*, travaux du XLIM et LISPEN). L'IA a également un fort impact sur la caractérisation de nuages de points, où la topologie est complexe à extraire, et que l'apprentissage permet d'inférer.

4 Transformation de formes, *levels set*, transport optimal

La communauté du GT MG a également beaucoup contribué au développement de méthodes de transport optimal pour la préservation de la matière, notamment pour le recalage de nuages de points, ou pour l'interpolation de formes, avec la contrainte de conservation de masse proposée par le transport (transport semi-discret symétrisé par exemple), avec le défaut de déchirement des formes induit par le transport qui vise à minimiser des distances de déplacement (INRIA Nancy, LIRIS). Plus récemment la communauté s'est emparé des problèmes de *morphing* entre formes à volume constant mais en se basant sur la *level set equation*, en formulation implicite neuronale permettant de garantir une advection à volume constant (LIRIS).

5 Représentations neuronales implicites

Les travaux sur les surfaces implicites avaient été nombreux avant les années 2000 (ICUBE, IRIT, LIRIS) et ont trouvé une seconde dynamique récemment. Dans la lignée des NeRF

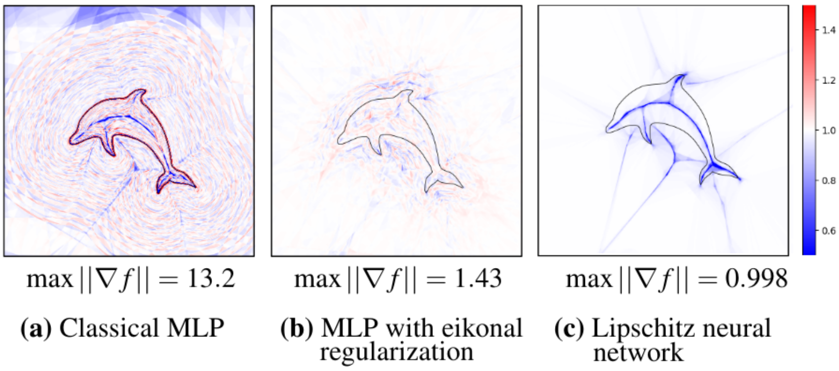


Figure 2 Les réseaux 1-Lipschitz garantissent que la fonction distance signée estimée est bien 1-lipschitz. Repris de [9].

est apparue (ou revenue) l'idée de paramétriser les fonctions implicites par des réseaux de neurones, c'est à dire d'optimiser des paramètres d'un réseau pour représenter, par exemple la fonction implicite d'une forme. Il est ensuite possible à la façon d'un *Physically Informed Neural Network* (PINN) de rajouter des contraintes d'EDP, comme par exemple l'équation eikonale qui doit être satisfaite par une fonction distance signée. Cette représentation est utile par exemple pour extraire des axes médians de manière robuste (LIRIS). On peut aussi contraindre par construction le réseau à produire des fonctions 1-lipschitz (INRIA Grenoble, voir Figure 2). Il est également possible de paramétriser temporellement cette fonction implicite pour modéliser des morphings de formes (LIRIS). Cela permet à nouveau d'intégrer des contraintes sous formes d'équations aux dérivées partielles et d'apprendre des champs d'advection avec certaines bonnes propriétés (voir au dessus).

6 Spécifications formelles et preuves en modélisation géométrique, contraintes

L'analyse qualitative des systèmes de contraintes utilise classiquement des méthodes combinatoires (le plus souvent des graphes, parfois des matroïdes). Cette analyse est indispensable, vu la taille des systèmes de contraintes, mais elle ne fonctionne bien que pour les systèmes bien contraints, que ce soit en 2D ou en 3D ; elle ne peut détecter que les erreurs les plus simples, dites structurelles, comme dans : $f(x, y, z) = g(z) = h(z) = 0$ qui sur-contraint l'inconnue z. En 3D, cette analyse se heurte à de sérieuses difficultés, comme la caractérisation des graphes rigides. Plus généralement, de nombreux théorèmes de géométrie ou de topologie provoquent des dépendances non structurelles entre les contraintes (ICUBE, LIB). Comme dans les autres domaines, il faut utiliser de manière systématique en modélisation géométrique des techniques avancées de spécifications formelles et de preuves de propriétés et de programmes, de préférence assistées par ordinateur. La résolution numérique (y compris par des méthodes d'analyse par intervalles) est guidée par l'analyse qualitative des systèmes de contraintes permettant de détecter les erreurs (sous- ou sur-contraintes) et de décomposer les systèmes bien contraints.

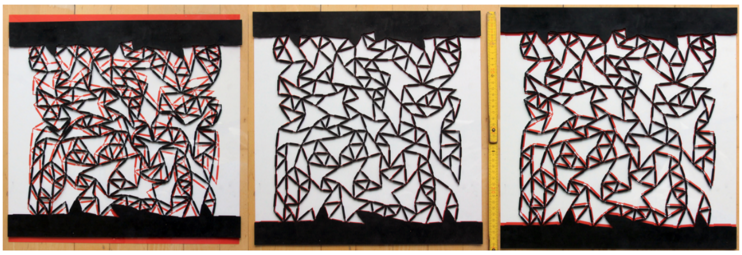


Figure 3 Optimisation géométrique pour la conception de méta-matériau auxetique. Repris de [2].

7 Analyse multirésolution, streaming et compression

Le lien entre topologie et géométrie est très fort, essentiel pour décrire une surface, des bords, des frontières. Les surfaces de subdivision ont longtemps tiré ce thème (IRIT, LIB). Leur facilité à raffiner une géométrie a fait de la compression une des applications de ces méthodes. La difficulté est alors de construire des maillages initiaux suffisamment légers mais dont la subdivision conduirait à un maillage dense identique à l'initial. Cela nécessite la maîtrise de subdivision inverse et le travail sur les schémas de subdivision (assurer la localité, la contraction, les contraintes de continuité...). Pour cela l'analyse multi-résolution et notamment les transformées en ondelettes sont toujours d'actualité.

Dorénavant, les méthodes NeRF et Gaussian Splat sont plus efficaces en terme de visualisation et s'attachent moins à la mesure de la géométrie (INRIA Sophia-Antipolis, IRIT).

Cette thématique porte également l'analyse de la rugosité des surfaces et la caractérisation des surfaces fractales (LIB). On représente le plus souvent des surfaces lisses par des surfaces de subdivision et des surfaces rugueuses par des surfaces fractales (IFS). Un formalisme commun (basé sur les IFS) a été mis en place (de manière théorique).

8 Structures géométriques

Un dernier axe de recherche actif dans le GT MG est l'optimisation de la géométrie pour l'impression 3D (Nancy, Grenoble) avec la prise en compte de caractéristiques souhaitées (compressibilité dans une direction par exemple). La figure 3 montre un exemple d'une telle optimisation pour un méta-matériau auxétique.

9 Logiciels et valorisation de la communauté MG

— membre du consortium CGAL : importante bibliothèque de géométrie algorithmique (LIRIS, INRIA Sophia-Antipolis)
— CGoGN, plateforme logicielle de modélisation géométrique et topologique de l'équipe IGG, basée sur CGAL (ICUBE)
— Jerboa, un modeleur géométrique à base de règles de transformation de graphes, modélisation basée sur la topologie, XLIM, Poitiers

Contributeurs et contributrices.

Julie Digne, Samuel Peltier, Romain Raffin.

Références

1 Dominique Attali, Mattéo Clémot, Bianca Dornelas, and André Lieutier. When alpha-complexes collapse onto codimension-1 submanifolds. In *Proceedings of the Symposium on Computational Geometry (SoCG)*, 2025.

2 Georges-Pierre Bonneau, Stefanie Hahmann, and Johana Marku. Geometric construction of auxetic metamaterials. *Computer Graphics Forum*, 40(2) :291–303, 2021.

3 Nicolas Bonneel and Julie Digne. A survey of optimal transport for computer graphics and computer vision. *Computer Graphics Forum*, 42(2) :439–460, 2023.

4 Alexander Braune, Mark Gillespie, Yiying Tong, and Mathieu Desbrun. Discrete torsion of connection forms on simplicial meshes. *ACM Transactions on Graphics (Proceedings of SIGGRAPH)*, 2025.

5 Anh Quoc Bui, Gilles Rougeron, Géraldine Morin, and Simone Gasparini. ROI-GS : Interest-based local quality 3D gaussian splatting. In *IEEE Visual Communications and Image Processing (VCIP)*, 2025.

6 Camille Buonomo, Julie Digne, and Raphaëlle Chaine. Volume preserving neural shape morphing. *Computer Graphics Forum (Proceedings of SGP)*, 44(5), 2025.

7 Mattéo Clémot and Julie Digne. Neural skeleton : Implicit neural representation away from the surface. *Computers & Graphics*, 114 :368–378, 2023.

8 Mattéo Clémot, Julie Digne, and Julien Tierny. Topological autoencoders++ : Fast and accurate cycle-aware dimensionality reduction. *IEEE Transactions on Visualization and Computer Graphics*, 32(2) :1622–1639, 2026.

9 Guillaume Coiffier and Louis Béthune. 1-Lipschitz neural distance fields. *Computer Graphics Forum (Proceedings of SGP)*, 43(5) :i–x, 2024.

10 Vincent Commin, Samuel Peltier, Arthur Cavalier, and Sébastien Horna. Clock mechanism generation using interaction graphs. *Computers & Graphics*, 2025.

11 Khoa Do, David Coeurjolly, Pooran Memari, and Nicolas Bonneel. Linear-time transport with rectified flows. *ACM Transactions on Graphics (Proceedings of SIGGRAPH)*, 2025.

12 Amine Farhat, Alexandre Bléron, Romain Vergne, and Joëlle Thollot. Motion ribbons : Parametrized surfaces for depicting motion effects. *Computers & Graphics*, 129 :104227, 2025.

13 Alex Fernandes, Steve Oudot, and François Petit. Computation of gamma-linear projected barcodes for multiparameter persistence. *Journal of Applied and Computational Topology*, 9(2) :12, 2025.

14 Vladimir Garanzha, Igor Kaporin, Liudmila Kudryavtseva, François Protais, Nicolas Ray, and Dmitry Sokolov. Foldover-free maps in 50 lines of code. *ACM Transactions on Graphics*, 40(4), 2021.

15 Guillaume Gisbert, Raphaëlle Chaine, and David Coeurjolly. Inpainting holes in folded fabric meshes. *Computers & Graphics*, 114 :201–209, 2023.

16 Diego Gomez, Bingchen Gong, and Maks Ovsjanikov. FourieRF : Few-shot NeRFs via progressive fourier frequency control. In *International Conference on 3D Vision (3DV)*, 2025.

17 Chems Eddine Himeur, Thibault Lejemble, Thomas Pellegrini, Mathias Paulin, Loïc Barthe, and Nicolas Mellado. PCEDNet : A lightweight neural network for fast and interactive edge detection in 3D point clouds. *ACM Transactions on Graphics*, 2021.

18 V. N. Huynh, H. H. Nguyen, and R. Raffin. A versatile multi-space DBSCAN framework for rough surface object segmentation. *Multimedia Tools and Applications*, 84 :39473–39497, 2025.

19 Maylis Jouvencel, Razmig Kéchichian, Julie Digne, and Sébastien Valette. SCONet : Convolutional occupancy networks for multi-organ segmentation. In *IEEE International Symposium on Biomedical Imaging - ISBI*, pages 1–5. IEEE, 2025.

20 Mathieu Ladeuil, Marc Trabucato, Alexis Vaisse, and Noura Faraj. Weighted feature graph via hierarchical clustering. In *Eurographics Posters*, 2025.

21 Vadim Lebovici, Jan-Paul Lerch, and Steve Oudot. Local characterization of block-decomposability for multiparameter persistence modules. *Homology, Homotopy and Applications*, 2025.

22 Léopold Maillard, Nicolas Sereyjol-Garros, Tom Durand, and Maks Ovsjanikov. DeBaRA : denoising-based 3D room arrangement generation. In *Advances in Neural Information Processing Systems (NeurIPS)*, 2024.

23 Niv Maruani, Yifan Wang, Matthew Fisher, Pierre Alliez, and Mathieu Desbrun. ShapeShifter. In *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition (CVPR)*, 2025.

24 Romain Pascual, Hakim Belhaouari, Agnès Arnould, and Pascale Le Gall. Inferring topological operations on generalized maps : Application to subdivision schemes. *Graphics and Visual Computing*, 6 :200049, 2022.

25 Samuel Peltier, Géraldine Morin, and Damien Aholou. Tubular parametric volume objects : Thickening a piecewise smooth 3D stick figure. *Computer Aided Geometric Design*, 85, 2021.

26 Mathieu Pietri, Eric Remy, Vincent Penné, and Jean-Luc Mari. Real-time live compression of dynamic 3D meshes for client-side GPU cloud gaming using skinning decomposition. *The Visual Computer*, 2025.

27 Clément Poull, Christian Gentil, Céline Roudet, Lucie Druoton, and Michael Roy. Second order differential properties of tensor product fractal surfaces. In *GRAPP*, 2025.

28 Yuang Shi, Simone Gasparini, Géraldine Morin, Chenggang Yang, and Wei Tsang Ooi. Sketch and patch : Efficient 3D Gaussian representation for man-made scenes. In *ACM Multimedia Systems Conference (MMVE)*, 2025.

29 Ramana Sundararaman, Nicolas Donati, Simone Melzi, Etienne Corman, and Maks Ovsjanikov. Deformation recovery : Localized learning for detail-preserving deformations. *ACM Transactions on Graphics (Proceedings of SIGGRAPH Asia)*, 2024.

30 Lucas Vergez, Arnaud Polette, and Jean-Philippe Pernot. Multi-part kinematic constraint prediction for automatic generation of CAD model assemblies using graph convolutional networks. *Computer-Aided Design*, 2025.

31 Giulio Viganò, Maks Ovsjanikov, and Simone Melzi. NAM : neural adjoint maps for refinement of shape correspondences. *ACM Transactions on Graphics (Proceedings of SIGGRAPH)*, 2025.

32 Chao Zhang, Arnaud Polette, Romain Pinquié, Gregorio Carasi, Henri De Charnace, and Jean-Philippe Pernot. eCAD-Net : Editable parametric cad models reconstruction from Dumb B-Rep models using deep neural networks. *Computer-Aided Design*, 178, 2025.